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sh(2/2) Superalgebra Eigenstates and Generalized Supercoherent and Supersqueezed States

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The superalgebra eigenstates (SAES) concept is introduced and then applied to find SAES associated to the sh(2/2) superalgebra, also known as Heisenberg–Weyl Lie superalgebra. This implies to solve a Grassmannian eigenvalue superequation. Thus, the sh(2/2) SAES contain the class of supercoherent states associated to the supersymmetric harmonic oscillator and also a class of supersqueezed states associated to the osp(2/2)Ð sh(2/2) superalgebra, where osp(2/2) denotes the orthosymplectic Lie superalgebra generated by the set of operators formed from the quadratic products of the Heisenberg–Weyl Lie superalgebra generators. The properties of these states are investigated and compared with those of the states obtained by applying the group-theoretical technics. Moreover, new classes of generalized supercoherent and supersqueezed states are also obtained. As an application, the super-Hermitian and η-pseudo-super-Hermitian Hamiltonians without a defined Grassmann parity and isospectral to the harmonic oscillator are constructed. Their eigenstates and associated supercoherent states are calculated.

KEY WORDS: superalgebra eigenstates; supercoherent; supersqueezed; Grassmann variables.

1. INTRODUCTION

The algebra eigenstates (AES) associated to a real Lie algebra have been defined as the set of eigenstates of an arbitrary complex linear combination of the generators of the considered algebra (Brif, 1996, 1997). According to the particular realization of the Lie algebra generators, the determination of AES implies, for instance, to solve an ordinary or a partial differential equation, to apply the operator technics, etc. For example, in the case of the su(2) Lie algebra, different approaches have been used such as the constellation formalism (Bacry, 1978),

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the ordinary first-order differential equations (Brif, 1997) or the operator method (Alvarez-Moraga, 2000). The same methods have also been applied to find AES for the su(1, 1) Lie algebra (Alvarez-Moraga, 2000; Brif, 1997). In the case of the two-photon AES, associated to the su(1, 1) Ð $h(2)$ Lie algebra, uses have been done of ordinary second-order differential equation (Brif, 1996). More recently, AES associated to the $h(2) \oplus su(2)$ Lie algebra have been obtained using these types of methods (Alvarez-Moraga and Hussin, 2002). In particular (Brif, 1997) it has been demonstrated that the generalized coherent states (GCS) associated to the SU(2) and SU(1, 1) Lie groups, based on group-theoretical approach (Perelomov, 1986), are subsets of the sets of AES associated to their corresponding Lie algebras. Moreover, the super coherent states of the supersymmetric harmonic oscillator (Cooper and Freed-Man, 1983; Salam and Strathdee, 1975; Salomonson and Van Holten, 1982; Wess and Zumino, 1971) as defined by Aragone and Zypman (1986) and a new class of supercoherent and supersqueezed states regarded as minimum uncertainty states have been obtained (Alvarez-Moraga and Hussin, 2002). Generalized supercoherent states (GSCS) associated to Lie supergroups have also been calculated following a generalized group-theoretical approach. This is the case, for example, of the supercoherent states associated to the following supergroups: Heisenberg–Weyl (H–W) and OSp(1/2) (Fatyga *et al.*, 1991) $U(1/2)$ (Hussin and Nieto, 1993; Sarkar, 1991), $U(1/1)$ (Pelezzola and Topi, 1992), and OSp(2/2) (El Gradechi and Nieto, 1996).

In the view of these approaches we ask the question of how we can generalize the AES concept valid for Lie algebras to Lie superalgebras. In general, as the even subspace of a Lie superalgebra is an ordinary Lie algebra, it is clear that the new concept must generalize in an appropriate form the AES concept. Indeed, the set of superalgebra eigenstates (SAES) associated to linear combinations of even generators of the Lie superalgebra must contain the AES associated to the Lie algebra generated by these generators. Moreover, we expect that the SAES associated to a certain class of superalgebras contain the GSCS of the related Lie supergroups. Another criterion to define the SAES concept starts from the utility that we can give to this concept when we study a particular quantum system, more precisely when we want to know the eigenstates of a physical observable represented by a super-Hermitian operator formed by a linear combination of the superalgebra generators or by a suitable product of these generators. According with these requirements, we propose the following definition of the SAES concept.

Definition 1.1. SAES associated to a Lie superalgebra correspond to the set of eigenstates of an arbitrary linear combination, with coefficients in the Grassmann algebra $\mathbb{C}B_L$, of the superalgebra generators. This means that if $\mathcal{L}$ is a superalgebra generated by the set of even operators $\Phi(a_1)$, $\Phi(a_2)$, ... $\Phi(a_m)$ and the set of odd operators $\Phi(a_{m+1})$, $\Phi(a_{m+2})$, ..., $\Phi_(a_{m+n})$, SAES associated to $\mathcal{L}$ are determined by the eigenvalue equation

$$\left[\sum_{i=1}^{m+n} B^i \Phi(a_i)\right] |\psi\rangle = Z|\psi\rangle, \tag{1}$$

where $B^i \in \mathbb{C}B_L, \forall i = 1, 2, \ldots, m+n$, and $Z \in \mathbb{C}B_L$.

In general, the superstate $|\psi\rangle$ is a linear combination, with coefficients in $\mathbb{C}B_L$, of the basis vectors of a graded super-Hilbert space $\mathcal{W}$, the representation space of the superalgebra on which it acts.

Let us here mention that Appendix A contains the notations and conventions used in the context of Grassmann algebras, Lie superalgebras, and supergroups. This will help for a good understanding of this work.

From the preceding definition, we see that to know explicitly SAES associated to a given Lie superalgebra, we must analyze case by case the different possible solutions of the Grassmannian eigenvalue equation (1), taking into account both the domain of definition of the Grassmann coefficients and the parity of them. In general, the calculations can be long and fastidious, but in physical applications, some simplifications appear because of some constraints on the coefficients like assuming a certain type of parity.

A natural generalization of the concept of AES to SAES starts with H–W superalgebra sh(2/2) generated by the bosonic operators $a, a^\dagger$, and I and the fermionic ones b and $b^\dagger$. We expect to recover the usual algebra eigenstates (Alvarez-Moraga and Hussin, 2002; Aragone and Zypman, 1986; Orszag and Salamo, 1988) but also supercoherent and supersqueezed states based on a group-theoretical approach (Kostelecký *et al.*, 1993; Nieto, 1992).

Let us remind that the well-known bosonic algebra is generated by the even operators $a, a^\dagger$, and I, which satisfy the usual nonzero commutation relation

$$[a, a^\dagger] = I, \tag{2}$$

and act on the usual Fock space $\mathcal{F}_b = \{|n\rangle, n \in \mathbb{N}\}$, as follows:

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad n \in \mathbb{N}. \tag{3}$$

The operators $a, a^\dagger$ are the usual annihilation and creation operators of the harmonic oscillator, and I acts as the identity operator. The corresponding fermionic superalgebra is generated by the odd operators $b, b^\dagger$ and the even operator I, which satisfy the nonzero supercommutation relation

$$\{b, b^\dagger\} = I. \tag{4}$$

These operators act on the graded space $\mathcal{F}_f = \{|+\rangle, |-\rangle\}$ as follows:

$$b|+\rangle = |-\rangle, \quad b|-\rangle = 0, \quad b^\dagger|+\rangle = 0, \quad b^\dagger|-\rangle = |+\rangle. \tag{5}$$

Taking the all set $\{a, a^\dagger, I, b, b^\dagger\}$ satisfying the nonzero supercommutation relations (2) and (4), we get the H–W superalgebra sh(2/2). Its acts naturally on the graded Fock space $\mathcal{F}_b \otimes \mathcal{F}_f = \{|n, \pm\rangle, n \in \mathbb{N}\}$. To compute SAES of this superalgebra, we will consider linear combinations over the field of Grassmann numbers. This means that, in general, we will deal with linear combinations of the bosonic (even) and fermionic (odd) operators with the coefficients taking values in the set $\mathbb{C}B_L$.

The paper will be thus distributed as follows. In section 2, we will determine SAES associated to the bosonic H–W Lie algebra. A significant difference with respect to the other approaches is now that linear combinations of generators is considered over the field of Grassmann numbers. Connections with preceding approaches will be made. In section 3, fermionic H–W Lie superalgebra will be considered. These special SAES cases will give a good understanding of the specificities induced by working with Grassmann-valued variables and will help us to give a complete description of SAES associated to the H–W Lie superalgebra in section 4. Finally, in section 5, Hamiltonians which are isospectral to the harmonic oscillator one will be constructed and their associated supercoherent states will be described. The notations and conventions used in this work will be revised in Appendix A, whereas the details of calculus of SAES of section 4 will be presented in Appendix B.

2. SAES ASSOCIATED TO THE HEISENBERG–WEYL LIE ALGEBRA, GENERALIZED SUPERCOHERENT, AND SUPERSQUEEZED STATES

SAES associated to the H–W Lie algebra will be obtained as the states $|\psi\rangle$ that verify the eigenvalue equation,

$$[A_- a + A_+ a^\dagger + A_3 I]|\psi\rangle = Z|\psi\rangle, \tag{6}$$

where $A_\pm$, A_3, and $Z \in \mathbb{C}B_L$. From the structure of this equation, we expect to recover the usual results concerning, in particular, the eigenstates of a, i.e., the standard coherent states of the harmonic oscillator (Perelomov, 1986). That is the reason why we begin our considerations by taking first $A_+ = A_3 = 0$. In this context, we will distinguish between the cases where $(A_-)_\phi$ is zero and not zero. Next, the general combination (6) will be considered with $(A_-)_\phi \neq 0$. This means that A_- is an invertible Grassmann number and the relation (6) thus reduces to

$$[a + \beta a^\dagger]\psi\rangle = z|\psi\rangle, \qquad \beta, z \in \mathbb{C}B_L. \tag{7}$$

2.1. Generalized Coherent States

If we take $A_+ = A_3 = 0$, the eigenvalue Equation (6) thus writes

$$A_{-a}|\psi\rangle = Z|\psi\rangle. \tag{8}$$

Let us assume a solution of the type

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \qquad C_n \in \mathbb{C}B_L. \tag{9}$$

By inserting (9) into (8), applying (3) and using the orthogonality property of states $\{|n\rangle\}_{n=0}^{\infty}$, we get to the following recurrence relation:

$$A_- C_{n+1} = \frac{Z C_n}{\sqrt{n+1}}, \qquad n = 0, 1, \ldots. \tag{10}$$

Here we must consider two cases: the cases $(A_-)_\phi \neq 0$ and $(A_-)_\phi = 0$.

In the first case, $(A_-)_\phi \neq 0$ is thus an invertible quantity and we can isolate the coefficient C_{n+1} in (10). It is easy to show that we get

$$C_n = \frac{((A_-)^{-1} Z)^n}{\sqrt{n!}} C_0, \qquad n = 1, 2, \ldots. \tag{11}$$

SAES associated to the operator $A_- a$ with eigenvalue Z are then given by

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} C_0 |n\rangle = \sum_{n=0}^{\infty} \frac{(z a^\dagger)^n}{n!} C_0 |0\rangle = e^{z a^\dagger} C_0 |0\rangle, \tag{12}$$

where $z = (A_-)^{-1} Z$. As we are interested in normalized eigenstates, we take $(C_0)_\phi \neq 0$ and the eigenstates can be written as

$$|z\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)|0\rangle, \tag{13}$$

where

$$\mathbb{D}(z_0) = \exp(z_0 a^\dagger - z_0^\ddagger a), \quad \mathbb{D}(z_1) = \exp(z_1 a^\dagger - z_1^\ddagger a), \tag{14}$$

$z_0 = ((A_-)^{-1} Z)_0$ and $z_1 = ((A_-)^{-1} Z)_1$.

We notice that the generalized coherent states associated to the harmonic oscillator system, considered as eigenstates of the annihilation operator a, are here given by (13) when $A_- = \epsilon_\phi$, i.e., when $z_0 = Z_0$ and $z_1 = Z_1$. These states are obtained by applying successively the superunitary operators $\mathbb{D}(Z_1)$ and $\mathbb{D}(Z_0)$ to the fundamental state $|0\rangle$.

In the second case, i.e., when $(A_-)_\phi = 0$, we cannot obtain a simple closed expression to describe all the algebra eigenstates. A class of solution is

$$C_n = \frac{(C_1 (C_0)^{-1})^n}{\sqrt{n!}} C_0, \qquad n = 2, 3, \ldots, \tag{15}$$

together with

$$A_- C_1 = Z C_0 \tag{16}$$

and C_0 is an arbitrary coefficient such that $(C_0)_\phi \neq 0$. The condition (16) implies that $Z_\phi = 0$ and is equivalent to the following system of superequations:

$$(A_-)_0(C_1)_0 + (A_-)_1(C_1)_1 = Z_0(C_0)_0 + Z_1(C_0)_1, \tag{17}$$

$$(A_-)_0(C_1)_1 + (A_-)_1(C_1)_0 = Z_0(C_0)_1 + Z_1(C_0)_0, \tag{18}$$

where we have decomposed A_-, C_0, C_1 and Z into their even and odd parts. This system can be solved to give C_1 in terms of C_0. A set of normalized eigenstates corresponding to the eigenvalue $Z = \alpha A_-$, $\alpha \in \mathbb{C}$, is given by the standard coherent states

$$|\alpha\epsilon_\phi\rangle = \exp(\alpha\epsilon_\phi a^\dagger - \bar{\alpha}\epsilon_\phi a)|0\rangle = \mathbb{D}(\alpha\epsilon_\phi)|0\rangle. \tag{19}$$

So, in the special case when $(A_-)_0 = 0$, the algebra eigenstates of the odd operator $(A_-)_1 a$ contain the set of coherent states of the standard harmonic oscillator.

2.1.1. Density of Algebra

It is interesting to mention that we can interpret this last result in terms of the concept of density of algebra. Indeed, let us define the odd operators

$$\mathbb{A}_- = z_1^\ddagger a, \quad \mathbb{A}_+ = -z_1 a, \qquad z_1 \in \mathbb{C}B_{L_1}. \tag{20}$$

By integrating these operators with respect to the corresponding odd variable, we get

$$a = \int \mathbb{A}_- dz_1^\ddagger, \qquad a^\dagger = \int dz_1 \mathbb{A}_+, \tag{21}$$

i.e., $\mathbb{A}_-$ and $\mathbb{A}_+$ fulfill the role of a linear density of the annihilation a and the creation $a^\dagger$, respectively. We notice that

$$[a, a^\dagger] = \int \{\mathbb{A}_-, \mathbb{A}_+\}\, dz_1^\dagger\, dz_1, \quad \{a, a^\dagger\} = \int [\mathbb{A}_-, \mathbb{A}_+]\, dz_1^\dagger\, dz_1, \tag{22}$$

i.e., the commutator and anticommutator of the even operators a and $a^\dagger$ are obtained by integrating, on the entire odd Grassmann space, the anticommutator and commutator of the odd operators $\mathbb{A}_-$ and $\mathbb{A}_+$, respectively. This suggests the following definitions of the densiy of identity $\mathbb{I}$ and of an energy type density $\mathbb{H}$:

$$\mathbb{I} = \{\mathbb{A}_-, \mathbb{A}_+\} = z_1 z_1^\ddagger, \qquad \mathbb{H} = [\mathbb{A}_-, \mathbb{A}_+] = \frac{w}{2} z_1 z_1^\ddagger \{a, a^\dagger\}. \tag{23}$$

As we know, the eigenstates of the annihilation operator corresponding to the complex eigenvalue α are given by the standard harmonic oscillator coherent states $|\alpha\rangle = D(\alpha)|0\rangle$. They verify the eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{24}$$

Multiplying both sides of this equation by $z_1^{\ddagger}$, and then integrating with respect to this Grassmann variable and finally using (21), we get

$$\int \mathbb{A}_{-}|\alpha\rangle\, dz_1^{\ddagger} = \int \alpha z_1^{\ddagger}|\alpha\rangle\, dz_1^{\ddagger}, \tag{25}$$

i.e., by comparing both sides of this last equation, we conclude that a class of eigenstates of the odd operator $\mathbb{A}_{-}$ corresponding to the $\alpha z_1^{\ddagger}$ eigenvalue are given by the standard harmonic oscillator coherent states $\epsilon_{\phi}|\alpha\rangle$.

2.2. Generalized Supersqueezed States

Let us now solve the eigenvalue Equation (7). A class of solutions can be constructed firstly, by expressing $|\psi\rangle$ in terms of a generalized su(1, 1) squeeze operator (the normalizer of the H–W algebra), following this way the construction of the standard squeezed states associated to the simple harmonic oscillator system (Orszag and Salamo, 1988). Indeed, let us write

$$|\psi\rangle = S(\mathcal{X}_0)|\varphi\rangle, \tag{26}$$

where the squeeze operator $S(\mathcal{X}_0)$ is given by

$$S(\mathcal{X}_0) = \exp\left(\mathcal{X}_0 \frac{(a^{\dagger})^2}{2} - \mathcal{X}_0^{\ddagger}\frac{a^2}{2}\right), \tag{27}$$

with $\mathcal{X}_0$ an even invertible Grassmann number, $\mathcal{X}_0^{\ddagger}$ its adjoint (see Appendix A).

Inserting (26) into (7), using the relation

$$S^{\ddagger}(\mathcal{X}_0) a S(\mathcal{X}_0) = \cosh(\|\mathcal{X}_0\|)a + \sqrt{\mathcal{X}_0}\big(\sqrt{\mathcal{X}_0^{\ddagger}}\big)^{-1}\sinh(\|\mathcal{X}_0\|)a^{\dagger}, \tag{28}$$

where $\|\mathcal{X}_0\| = \sqrt{\mathcal{X}_0\mathcal{X}_0^{\ddagger}}$ and choosing $\mathcal{X}_0$ in such a way that it satisfies

$$\sqrt{\mathcal{X}_0}\big(\sqrt{\mathcal{X}_0^{\ddagger}}\big)^{-1}\sinh(\|)\mathcal{X}_0\|) + \beta_0\cosh(\|)\mathcal{X}_0\|) = 0, \tag{29}$$

we get the following eigenvalue equation for $|\varphi\rangle$:

$$[\mathcal{G}(\mathcal{X}_0, \beta)a + \beta_1\cosh(\|\mathcal{X}_0\|)a^{\dagger}]\varphi\rangle = z|\varphi\rangle, \tag{30}$$

where

$$\mathcal{G}(\mathcal{X}_0, \beta) = \cosh(\|\mathcal{X}_0\|) + \beta\sqrt{\mathcal{X}_0^{\ddagger}}\big(\sqrt{\mathcal{X}_0}\big)^{-1}\sinh(\|\mathcal{X}_0\|). \tag{31}$$

Let us notice that this last coefficient can be written on the form

$$\mathcal{G}(\mathcal{X}_0, \beta) = \mathcal{G}(\mathcal{X}_0, \beta_0)\big(\epsilon_{\phi} + \beta_1(\mathcal{G}(\mathcal{X}_0, \beta_0))^{-1}\sqrt{\mathcal{X}_0^{\ddagger}}(\sqrt{\mathcal{X}_0})^{-1}\sinh(\|\mathcal{X}_0\|)\big) \tag{32}$$

where, taking into account (29),

$$\mathcal{G}(\mathcal{X}_0, \beta_0) = \left[\epsilon_\phi - \beta_0^2 \mathcal{X}_0^\ddagger (\mathcal{X}_0)^{-1}\right] \cosh(\|\mathcal{X}_0\|). \tag{33}$$

Multiplying both sides of Eq. (30) by the inverse of $\mathcal{G}(\mathcal{X}_0, \beta)$ and taking into account (32), we get

$$[a + \hat{\beta}_1 a^\dagger]|\varphi\rangle = \hat{z}|\varphi\rangle, \tag{34}$$

where

$$\hat{\beta}_1 = \beta_1 (\mathcal{G}(\mathcal{X}_0, \beta_0))^{-1} \cosh(\|\mathcal{X}_0\|) \in \mathbb{C}B_{L_1}, \tag{35}$$

and

$$\hat{z} = \left[(\mathcal{G}(\mathcal{X}_0, \beta_0))^{-1} - \beta_1 \sqrt{(\mathcal{X}_0^\ddagger)}\left(\sqrt{\mathcal{X}_0}\right)^{-1} \sinh(\|\mathcal{X}_0\|)\right] z. \tag{36}$$

Equation (34) is thus simpler to solve than (7). Indeed, we can again try a solution of the type

$$|\varphi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \qquad C_n \in \mathbb{C}B_L. \tag{37}$$

Inserting it into (34), using the raising and lowering properties of the operators $a^\dagger$ and a, and the orthogonality conditions of the states $\{|n\rangle\}$, we get the recurrence relation

$$C_{n+1} = \frac{\left[\hat{z} C_n - \sqrt{n}\hat{\beta}_1 C_{n-1}\right]}{\sqrt{n+1}}, \qquad n = 2, \ldots, \tag{38}$$

with

$$C_1 = \hat{z} C_0, \tag{39}$$

and C_0 is an arbitrary constant. Proceeding by iteration, we get

$$C_n = \frac{1}{\sqrt{n!}} \left(\hat{z}^n - \sum_{k=0}^{n-2} (k+1) \hat{z}^{(n-2-k)} (\hat{z}^*)^k \hat{\beta}_1 \right) C_0, \quad n = 2, 3, \ldots. \tag{40}$$

This expression may be written in a closed form. Indeed, as we can show that

$$\sum_{k=0}^{n-2} (k+1) \hat{z}^{(n-2-k)} (\hat{z}^*)^k = \frac{n(n-1)}{2!} (\hat{z}_0)^{n-2} - \frac{n(n-1)(n-2)}{3!} (\hat{z}_0)^{n-3}\, \hat{z}_1 \tag{41}$$

$$= \frac{1}{2!} \frac{\partial^2}{\partial \hat{z}_0^2} (\hat{z}_0)^n - \frac{1}{3!} \frac{\partial^3}{\partial \hat{z}_0^3} (\hat{z}_0)^n\, \hat{z}_1, \tag{42}$$

the relation (40) becomes

$$C_n = \frac{1}{\sqrt{n!}}\left(\hat{z}^n - \left[\frac{1}{2!}\frac{\partial^2}{\partial \hat{z}_0^2}(\hat{z}_0)^n - \frac{1}{3!}\frac{\partial^3}{\partial \hat{z}_0^3}(\hat{z}_0)^n\,\hat{z}_1\right]\hat{\beta}_1\right)C_0, \qquad n = 2, 3, \ldots, \tag{43}$$

which is also valid for $n = 1$. Finally, inserting this result into (37), and after some manipulations, we obtain a general solution of (34), which is

$$|\varphi\rangle = \left[e^{\hat{z}_1 a^\dagger} - \hat{\beta}_1\frac{(a^\dagger)^2}{2!} + \hat{z}_1\hat{\beta}_1\frac{(a^\dagger)^3}{3!}\right]e^{\hat{z}_0 a^\dagger}|0\rangle C_0. \tag{44}$$

A normalized version of (44) is given by

$$|\varphi\rangle = \exp\left[-\hat{\beta}_1\frac{(a^\dagger)^2}{2} - \hat{z}_1\hat{\beta}_1\frac{(a^\dagger)^3}{3}\right]\mathbb{D}(\hat{z}_0)\mathbb{D}(\hat{z}_1)|0\rangle\hat{C}(\hat{z}, \hat{\beta}_1), \tag{45}$$

where the operator $\mathbb{D}$ has been defined in (14). The normalization constant $\hat{C}$ is given by

$$\hat{C}(\hat{z}, \hat{\beta}_1) = (\sqrt{\Gamma})^{-1}\left[\epsilon_\phi + \frac{1}{2}(\sqrt{\Gamma})^{-1}\Omega(\sqrt{\Gamma})^{-1}\right], \tag{46}$$

with

$$\Gamma(\hat{z}, \hat{\beta}_1) = \epsilon_\phi - \frac{1}{2}((\hat{z}^\ddagger)^2\hat{\beta}_1 + (\hat{\beta}_1)^\ddagger\hat{z}^2) - \frac{1}{3}((\hat{z}^\ddagger)^3\hat{z}_1\hat{\beta}_1 + (\hat{\beta}_1)^\ddagger(\hat{z}_1)^\dagger\hat{z}^3) \tag{47}$$

and

$$\begin{aligned}\Omega(\hat{z}, \hat{\beta}_1) = &\left[\frac{1}{6}((\hat{z}^\ddagger)^3\hat{z}_1\hat{z}^2 + (\hat{z}^\ddagger)^2(\hat{z}_1)^\ddagger\hat{z}^3)\right.\\ &+((\hat{z}^\ddagger)^2\hat{z}_1\hat{z} + (\hat{z}^\ddagger)\hat{z}_1 + (\hat{z}^\ddagger)(\hat{z}_1)^\ddagger\hat{z}^2 + (\hat{z}_1)^\ddagger\hat{z})\\ &\left.-\frac{1}{4}((\hat{z}^\ddagger)^2\hat{z}^2 + 4\hat{z}^\ddagger\hat{z} + 2)\right](\hat{\beta}_1)^\ddagger\hat{\beta}_1\\ &-\frac{1}{9}((\hat{z}^\ddagger)^3\hat{z}^3 + 9(\hat{z}^\ddagger)^2\hat{z}^2 + 24\hat{z}^\ddagger\hat{z} + 6)(\hat{z}_1)^\ddagger\hat{z}_1(\hat{\beta}_1)^\ddagger\hat{\beta}_1\\ &-((\hat{z}_0^\ddagger)^2\hat{\beta}_1 + (\hat{\beta}_1)^\ddagger(\hat{z}_0)^2)(\hat{z}_1)^\ddagger\hat{z}_1.\end{aligned}$$

From (26) and (45) we conclude that a class of normalized solutions of the eigenvalue equation (7), corresponding to the eigenvalue z, is given by the generalized supersqueezed states

$$|\psi\rangle = S(\mathcal{X}_0)\exp\left[-\hat{\beta}_1\frac{(a^\dagger)^2}{2} - \hat{z}_1\hat{\beta}_1\frac{(a^\dagger)^3}{3}\right]\mathbb{D}(\hat{z}_0)\mathbb{D}(\hat{z}_1)|0\rangle\hat{C}(\hat{z}, \hat{\beta}_1). \tag{48}$$

Let us now give some examples of such states.

2.2.1. *Standard Supersqueezed States*

The standard supersqueezed states are obtained from (48) when $\beta_1 = 0$ and $z_1 = 0$, i.e., when $\hat{\beta}_1 = 0$, $\hat{z}_1 = 0$, and $\hat{z}_0 = (\mathcal{G}(\mathcal{X}_0, \beta_0))^{-1} z_0$. They are given by

$$|\psi\rangle = S(\mathcal{X}_0)\mathbb{D}(\hat{z}_0)|0\rangle, \tag{49}$$

where $\mathcal{X}_0$ and $\hat{z}_0$ remain even Grassmann-valued numbers.

2.2.2. *A New Class of Supersqueezed States*

Another class of supersqueezed states appears in (48), because of the possibility to choose in (7) a nonzero odd component of the variable β. For example, if we choose $\beta_0 = 0$, i.e., $\mathcal{X}_0 = 0$, $\hat{\beta}_1 = \beta_1$, and $\hat{z} = z$, then from (48) we obtain the following class of states:

$$|\psi\rangle = \exp\left[-\beta_1 \frac{(a^\dagger)^2}{2} - z_1\beta_1 \frac{(a^\dagger)^3}{3}\right]\mathbb{D}(z_0)\mathbb{D}(z_1)|0\rangle \hat{C}(z, \beta_1). \tag{50}$$

They are obtained by applying the operator

$$\exp\left[-\beta_1 \frac{(a^\dagger)^2}{2} - z_1\beta_1 \frac{(a^\dagger)^3}{3}\right] \tag{51}$$

to the generalized coherent states (13) of a. In the special case where $z_1 = 0$, we get to the normalized supersqueezed states

$$|\psi\rangle = \left[\epsilon_\phi + \frac{1}{4}\beta_1^\ddagger\beta_1\left(z_0^2(z_0^\ddagger)^2 + 4z_0 z_0^\ddagger + 2\right)\right]\exp\left[-\frac{1}{8}\beta_1^\ddagger\beta_1(a^2(a^\dagger)^2 + (a^\dagger)^2 a^2)\right]$$
$$\times \exp\left[-\left(\beta_1 \frac{(a^\dagger)^2}{2} - \beta_1^\ddagger \frac{a^2}{2}\right)\right]\mathbb{D}(z_0)|0\rangle, \tag{52}$$

which are written in terms of the superunitary operator $S(-\beta_1)$ as defined in (27). Moreover, in the case where $\beta_1 \in \mathbb{R}B_{L_1}$, this last equation becomes

$$|\psi\rangle = S(-\beta_1)\mathbb{D}(z_0)|0\rangle, \tag{53}$$

i.e., we are in the presence of a class of supersqueezed states which are constructed by applying the superunitary supersqueeze operator $S(-\beta_1)$ to the standard harmonic oscillator coherent states.

3. SAES ASSOCIATED TO THE FERMIONIC SUPERALGEBRA

In this section, we will construct SAES associated with the fermionic superalgebra generated by $\{b, b^\dagger, I\}$ which satisfy the nonzero supercommutation relation (4). The general eigenvalue equation writes as

$$[B_- b + B_+ b^\dagger + B_3 I]|\psi\rangle = Z|\psi\rangle, \qquad B_\pm, Z \in \mathbb{C}B_L. \tag{54}$$

Here we will distinguish again two cases: firstly when $B_+ = B_3 = 0$ and secondly when B_- is invertible so that Eq. (54) reduces to

$$(b + \delta b^\dagger)|\psi\rangle = z|\psi\rangle, \qquad \delta, z \in \mathbb{C}B_L. \tag{55}$$

3.1. The *b*-Fermionic Eigenstates

Let us solve

$$Bb|\psi\rangle = Z|\psi\rangle, \qquad B, Z \in \mathbb{C}B_L. \tag{56}$$

Since the fermionic graded Fock space is reduced to the vectors $|-\rangle$ (even) and $|+\rangle$ (odd) which act as in (5), a solution of (56) writes as

$$|\psi\rangle = C|-\rangle + D|+\rangle, \qquad C, D \in \mathbb{C}B_L. \tag{57}$$

Inserting (57) into (56) and using (5), we get

$$BD^*|-\rangle = ZC|-\rangle + ZD|+\rangle. \tag{58}$$

The orthogonality of the states $|-\rangle$ and $|+\rangle$ leads to the following set of algebraic equations:

$$BD^* = ZC, \quad ZD = 0, \tag{59}$$

or by conjugation of the first one,

$$B^*D = Z^*C^*, \quad ZD = 0. \tag{60}$$

Let us mention that, when $B_\phi = 0$, we have evidently the normalized solution $|\psi\rangle = |-\rangle$ when the eigenvalue Z is zero, but because of the presence of Grassmann value quantities, when $B_\phi = 0$, we have a larger set of solutions. For instance, for $B = B_1$, we find a solution of the form

$$|\psi\rangle = C|-\rangle \pm B_1|+\rangle. \tag{61}$$

Normalized eigenstates are given by

$$|\psi\rangle = \exp[\pm(B_1 b^\dagger + B_1^\ddagger b)]|-\rangle. \tag{62}$$

When $Z \neq 0$, nontrivial solutions appear if and only if $Z_\phi = 0$. From (60), we have $D_\phi = 0$. To solve completely the system (60), we have to distinguish two cases.

If $B_\phi \neq 0$, we can solve D from the first equation of (60)

$$D = (B^*)^{-1}Z^*C^* = (B^{-1}Z)^*C^* = z^*C^*, \tag{63}$$

where $z = z_0 + z_1 = (B^{-1}Z)$. Now inserting (63) into the second equation of (60), we get

$$Zz^*C^* = 0. \tag{64}$$

Normalized solutions will be obtained if $C_\phi \neq 0$ and we thus get

$$Zz^* = 0, \tag{65}$$

which can be written explicitly

$$z_0^2 = 0, \qquad z_0 Z_1 = z_1 Z_0. \tag{66}$$

The normalized eigenstates of Bb with the eigenvalue Z satisfying (66) are given by

$$|\psi\rangle = |-\rangle + z^*|+\rangle)C, \tag{67}$$

where C is an arbitrary Grassmann number such that $C_\phi \neq 0$. They can be written as

$$|z_0; z_1\rangle = \mathbb{T}(z_1)\mathbb{T}(z_0)|-\rangle, \tag{68}$$

where the superunitary operators $\mathbb{T}$ are given by

$$\mathbb{T}(z_1) = \exp(b^\dagger z_1 - z_1^\ddagger b), \quad \mathbb{T}(z_0) = \exp(z_0 b^\dagger - z_0^\ddagger b). \tag{69}$$

The b-SAES are obtained from (68) when $B = \epsilon_\phi$, so that $z_0 = Z_0$ and $z_1 = Z_1$. We notice that when $z_0 = 0$, they reduce to the standard supercoherent states associated to the system characterized by the fermionic Hamiltonian $H = b^\dagger b - \frac{1}{2}$.

If $B_\phi = 0$, the problem is a little more tricky. We can write (59) explicitly as

$$B_0 d_0 - B_1 d_1 = Z_0 c_0 + Z_1 c_1, \tag{70}$$

$$B_1 d_0 - B_0 d_1 = Z_1 c_0 + Z_0 c_1, \tag{71}$$

$$Z_0 d_0 + Z_1 d_1 = 0, \tag{72}$$

$$Z_1 d_0 + Z_0 d_1 = 0, \tag{73}$$

where we have taken $C = c_0 + c_1$ and $D = d_0 + d_1$. In this way, for instance, when $B_0 \neq 0$ and $(B_0)^2 \neq 0$, we can combine (70) and (71) to obtain

$$(B_0)^2 d_0 = (B_0 Z_0 - B_1 Z_1)c_0 + (B_0 Z_1 - B_1 Z_0)c_1, \tag{74}$$

$$(B_0)^2 d_1 = (B_1 Z_0 - B_0 Z_1)c_0 + (B_1 Z_1 - B_0 Z_0)c_1, \tag{75}$$

and then combine this last system of equations with (72) and (73) to get

$$Z_0(2B_1 Z_1 - B_0 Z_0)c_0 + B_1(Z_0)^2 c_1 = 0, \tag{76}$$

$$Z_0(2B_1 Z_1 - B_0 Z_0)c_1 + B_1(Z_0)^2 c_0 = 0. \tag{77}$$

The systems (74 and 75) and (76 and 77) are equivalent to

$$(B_0)^2 D = BZ^* C^* \tag{78}$$

and

$$Z_0(2B_1 Z_1 - B_0 Z_0 + B_1 Z_0)C = 0, \tag{79}$$

respectively. As we search for normalized solutions, we must take $C_\phi \neq 0$. This implies the following condition for the Z eigenvalue:

$$Z_0(2B_1 Z_1 - B_0 Z_0) = 0, \tag{80}$$

$$B_1(Z_0)^2 = 0. \tag{81}$$

Then, the normalized eigenstates of (56) corresponding to the Z eigenvalue satisfying (80 and 81) are given by (57), with C an arbitary Grassmann number such that $C_\phi \neq 0$, and D verifying (78).

Following a similar procedure, when $B_0 = 0$ and $B_1 \neq 0$, the normalized solutions of (56) corresponding to the Z eigenvalue satisfying the conditions:

$$(Z_0)^2 = 0, \qquad Z_0 Z_1 = 0, \tag{82}$$

are given by (57), with $C_\phi \neq 0$, and D verifying

$$B_1 D = -Z^* C^*. \tag{83}$$

When $B_0 \neq 0$ and $B_1 = 0$, the solutions corresponding to the Z eigenvalue satisfying the conditions

$$(Z_0)^2 = 0, \tag{84}$$

are given by (57), with $C_\phi \neq 0$, and D verifying

$$B_0 D = Z^* C^*. \tag{85}$$

Other classes of solutions can be reached by imposing other conditions on the coefficient B.

3.2. Supersqueezed States

Let us now solve the eigenvalue (55). If we assume again a solution of the type (57), then by inserting it into (55), using the raising and lowering properties (5) and the orthogonality between the sates $|-\rangle$ and $|+\rangle$, we get the following algebraic Grassmann equations for determining C and D:

$$D^* = zC, \tag{86}$$

$$\delta C^* = zD. \tag{87}$$

By conjugating Eq. (86) and then by inserting it into (87), we get

$$(zz^* - \delta)C^* = 0. \tag{88}$$

As we are interested in normalized solutions, we must take $C_\phi \neq 0$, and then (88) implies

$$z_0^2 = \delta, \tag{89}$$

i.e., δ is an even Grassmann number. Inserting (86) into (57) and considering the conditions (89), we conclude that a set of normalized eigenstates of the operator $(b + \delta_0 b^\dagger)$ corresponding to the eigenvalue $z = \pm\sqrt{\delta_0} + z_1$ is given by

$$|\delta_0, z_1\rangle^\pm = \big(|-\rangle - (z_1 \mp \sqrt{\delta_0})|+\rangle\big)C. \tag{90}$$

It is not too hard to show that the corresponding normalized supersqueezed states are given by

$$|\delta_0, z_1\rangle^\pm = \exp\big(b^\dagger z_1 - z_1^\ddagger b\big)\exp[\pm\sqrt{\delta_0}(b^\dagger + z_1^\ddagger)]|-\rangle N^\pm(\delta_0, z_1), \tag{91}$$

where the normalization constant $N^\pm$ is given by

$$N^\pm(\delta_0, z_1) = \mathcal{F}^{-1}\left[\epsilon_\phi \mp \frac{1}{2}\mathcal{F}^{-1}(\sqrt{\delta_0}z_1^\ddagger + (\sqrt{\delta_0})^\ddagger z_1 \mp \sqrt{\delta_0}(\sqrt{\delta_0})^\ddagger z_1^\ddagger z_1)\mathcal{F}^{-1}\right], \tag{92}$$

with

$$\mathcal{F}(\delta_0) = \sqrt{1 + \sqrt{\delta_0}(\sqrt{\delta_0})^\ddagger}. \tag{93}$$

We notice that in the limit $\delta_0 \mapsto 0$ the supersqueezed states (91) become the eigenstates of the operator b corresponding to the eigenvalue $z = z_1$.

4. SAES ASSOCIATED TO THE HEISENBERG–WEYL LIE SUPERALGEBRA

Let us now compute SAES associated to the H–W Lie superalgebra generated by the set of generators $\{a, a^\dagger, I, b, b^\dagger\}$ whose nonzero super-commutation relations are given by the relations (2) and (4). The eigenvalue equation is written as

$$[A_- a + A_+ a^\dagger + A_3 I + B_- b + B_+ b^\dagger]|\psi\rangle = Z|\psi\rangle,\ A_\pm, A_3, B_\pm, Z \in \mathbb{C}B_L. \tag{94}$$

Here we concentrate in the case where $(A_-)_\phi \neq 0$, i.e., A_- is an invertible Grassmann number. In this case, we can express (94) in the form

$$[a + \beta a^\dagger + \gamma b + \delta b^\dagger]|\psi\rangle = z|\psi\rangle, \quad \beta, \gamma, \delta, z \in \mathbb{C}B_L. \tag{95}$$

Special cases of this problem have been considered in sections 2 and 3. Here we consider the cases where we have the presence of both bosonic and fermionic operators in the eigenvalue Eq. (95).

4.1. Generalized Supercoherent States

First, we take the particular eigenvalue equation

$$[a + \gamma b]|\psi\rangle = z|\psi\rangle, \qquad \gamma, z \in \mathbb{C}B_L. \tag{96}$$

Let us assume a solution of the type

$$|\psi\rangle = \sum_{n=0}^{\infty}(C_n|n; -\rangle + D_n|n; +\rangle), \tag{97}$$

where $C_n, D_n \in \mathbb{C}B_L$. By inserting (97) in (96), using the lowering properties of operators a and b, Eqs. (3) and (5), and the orthogonality properties of the graded Fock space basis $\{|n; -\rangle, |n; +\rangle, n \in \mathbb{N}\}$, we get the recurrence relations

$$\sqrt{n+1}C_{n+1} + \gamma D_n^* = zCn, \tag{98}$$

$$\sqrt{n+1}D_{n+1} = zD_n. \tag{99}$$

From (99), it is easy to find the expression of the coefficients D_n in terms of an arbitrary constant D_0:

$$D_n = \frac{z^n}{\sqrt{n!}}D_0, \qquad n = 1, 2, \ldots. \tag{100}$$

Then, by inserting (100) in (98), we get the following recurrence relation for the coefficients C_n:

$$C_{n+1} = \frac{1}{\sqrt{n+1}}\left[zC_n - \gamma\frac{(z^*)^n}{\sqrt{n!}}D_0^*\right], \qquad n = 0, 1, 2, \ldots. \tag{101}$$

Finally, proceeding by iteration we get

$$C_n = \frac{1}{\sqrt{n!}}\left[z^nC_0 - \left(\sum_{k=0}^{n-1} z^{(n-1-k)}\gamma(z^*)^k\right)D_0^*\right], \qquad n = 1, 2, \ldots, \tag{102}$$

where C_0 is an arbitrary constant. Sicne C_0 and D_0 are arbitrary constants, Eq. (97) gives two independent solutions. The first one consists of the standard coherent states

$$|z; -\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}C_0|n; -\rangle. \tag{103}$$

To find the second one, we use the formula

$$\frac{1}{n+1}\sum_{k=0}^{n} z^{(n-k)}\gamma(z^*)^k = \left(\gamma_0 z_0^n + z^n\gamma_1\right). \tag{104}$$

We thus get the generalized coherent states on the form

$$|\widetilde{z,\gamma;+}\rangle = |z,\widetilde{\gamma_0,\gamma_1};+\rangle = \left[\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n}!}|n;+\rangle - a^\dagger\sum_{n=0}^{\infty}\frac{\left(\gamma_0 z_0^n + z^n\gamma_1\right)}{\sqrt{n}!}|n;-\rangle\right] D_0^*$$

$$= \exp[-(\gamma_0(1+z_1 a^\dagger)+\gamma_1)a^\dagger b]e^{za^\dagger}|0;+\rangle D_0^*. \tag{105}$$

The normalized version of the states (103) is given by

$$|z;-\rangle = |z_0,z_1;-\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)|0;-\rangle. \tag{106}$$

It is similar to the one obtained in (13). A set of normalized generalized supercoherent states, orthogonal to (106), is given by the formula

$$|z,\gamma,+\rangle = |z_0,z_1,\gamma_0,\gamma_1;+\rangle = \frac{|z,\widetilde{\gamma_0,\gamma_1},+\rangle - |z;-\rangle\langle -;z|z,\widetilde{\gamma_0,\gamma_1},+\rangle}{\||z,\widetilde{\gamma_0,\gamma_1},+\rangle - |z;-\rangle\langle -;z|z,\widetilde{\gamma_0,\gamma_1},+\rangle\|}. \tag{107}$$

After some claculations, we get the set of generalized supercoherent states

$$|z_0,z_1,\gamma_0,\gamma_1;+\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)\Bigg\{|0;+\rangle - \left[\left(1-\frac{1}{2}z_1^\ddagger z_1\right)\mathbb{D}(-z_1)(a^\dagger + z_0^\ddagger)\gamma_0 e^{z_1 z_0^\ddagger} + (1+z_1^\ddagger z_1)a^\dagger\gamma_1 - (1-z_1^\dagger z_1)z^\ddagger\gamma_0 e^{z_1 z_0^\ddagger}\right]|0;-\rangle\Bigg\}N(z_0,z_1,\gamma_0,\gamma_1), \tag{108}$$

where the normalization constant N is given by

$$N(z_0,z_1,\gamma_0,\gamma_1) = \mathcal{B}^{-1}\left[1-\mathcal{B}^{-1}\left(\gamma_1^\ddagger\gamma_1 - \gamma_0^\ddagger\gamma_0\left(z_0^\ddagger z_0\right)^2\right)z_1^\ddagger z_1\mathcal{B}^{-1}\right], \tag{109}$$

with

$$\mathcal{B}(\gamma_0,\gamma_1) = \sqrt{1+\gamma^\ddagger\gamma} = \sqrt{1+\gamma_0^\ddagger\gamma_0+\gamma_0^\ddagger\gamma_1+\gamma_1^\ddagger\gamma_0+\gamma_1^\ddagger\gamma_1}. \tag{110}$$

4.1.1. Supercoherent States

The supercoherent states (108) constitute a generalization of the supercoherent states found by Aragone and Zypman (1986). Indeed, from Eqs. (108–110) we see that, in the case where $\gamma_1 = 0$ and $z_1 = 0$, we have

$$|z_0,0,\gamma_0,0;+\rangle = \left(\sqrt{1+\gamma_0^\ddagger\gamma_0}\right)^{-1}\mathbb{D}(z_0)(|0;+\rangle - \gamma_0 a^\dagger|0;-\rangle). \tag{111}$$

4.1.2. Other Classes of Supercoherent States

Now if in (108–110), we take $\gamma_0 = 0$ and $z_0 = 0$, we get

$$|0, z_1, 0, \gamma_1; +\rangle = \left(1 - \frac{1}{2}\gamma_1^{\ddagger}\gamma_1 - \gamma_1^{\ddagger}\gamma_1 z_1^{\ddagger} z_1\right)\mathbb{D}(z_1) \times \left(|0; +\rangle - (1 + z_1^{\ddagger} z_1\right)a^{\dagger}\gamma_1|0; -\rangle). \quad (112)$$

We can also distinguish the case where $\gamma_1 = 0$ and $z_0 = 0$. We get

$$|0, z_1, \gamma_0, 0; +\rangle = \left(\sqrt{1 + \gamma_0^{\ddagger}\gamma_0}\right)^{-1}\mathbb{D}(z_1) \left\{|0; +\rangle + \gamma_0\left[\left(\frac{z_1^{\ddagger} z_1}{2} - 1\right)\mathbb{D}(-z_1)a^{\dagger} + z_1^{\ddagger}\right]|0; -\rangle\right\}. \quad (113)$$

4.1.3. Standard Supercoherent States

In the case where $\gamma = 0$, (108) becomes the standard coherent states

$$|z; +\rangle = |z_0, z_1; +\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)|0; +\rangle. \quad (114)$$

By combining the two independent solutions (106) and (114), we can construct a solution of the type

$$|z; \rho, \tau\rangle = \rho|z; -\rangle + \tau|z; +\rangle, \quad (115)$$

where ρ and τ are Grassmann numbers such that $\rho_1 z_1 = \tau_1 z_1 = 0$. Thus the states (115) are eigenstates of a corresponding to the eigenvalue z. In particular, if we take, for example, $\rho = 1 - \frac{z_1^{\ddagger} z_1}{2}$ and $\tau = -z_1$, then we obtain the supercoherent states

$$|z\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)\mathbb{T}(z_1)|0; -\rangle. \quad (116)$$

Moreover, if we take $z_1 = 0$, $\rho = 1 - \frac{\theta_1^{\ddagger}\theta_1}{2}$ and $\tau = -\theta_1$, we get the standard supercoherent states associated to the supersymmetric harmonic oscillator (Bérubé-Lauzière and Hussin, 1993; Fatyga *et al.*, 1991)

$$|z_0, \theta_1\rangle = \mathbb{D}(z_0)\mathbb{T}(\theta_1)|0; -\rangle. \quad (117)$$

4.2. Generalized Supersqueezed States

Let us now find SAES associated to the sub-superalgebra $\{a, b, b^{\dagger}, I\}$. If the coefficient of a in the linear combination is invertible the problem reduces to solve the eigenvalue equation:

$$[a + \gamma b + \delta b^{\dagger}]|\psi\rangle = z|\psi\rangle, \quad \gamma, \delta \in \mathbb{C}B_L. \quad (118)$$

We can show (see Appendix B, section SAES of $a + \gamma b + \delta b^{\dagger}$), that two classes of independent solutions of the eigenvalue Eq. (118) exist and are given by

$$|\psi; -\rangle = \left[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{a^{\dagger}}(\ell, \gamma, \delta^{*}, z_1)\, e^{z a^{\dagger}} |0; -\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{a^{\dagger}}(\ell, \delta, \gamma^{*}, z_1)\, e^{z a^{\dagger}} |0; +\rangle \right] C_0 \tag{119}$$

and

$$|\psi; +\rangle = \left[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{a^{\dagger}}(\ell, \gamma, \delta^{*}, z_1)\, e^{z a^{\dagger}} |0; +\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{a^{\dagger}}(\ell, \delta, \gamma^{*}, z_1)\, e^{z a^{\dagger}} |0; -\rangle \right] D_0^{*}, \tag{120}$$

where C_0 and D_0^{*} are arbitrary and invertible Grassmann constants and

$$\begin{aligned}\mathcal{O}_{a^{\dagger}}(\ell, \gamma, \delta^{*}, z_1) = \frac{1}{\ell!} \Bigg\{ & \overbrace{(\gamma\delta^{*}\gamma\delta^{*}\cdots)}^{\ell \text{ factors}} ((a^{\dagger})^{\ell} - z_1 (a^{\dagger})^{\ell+1}) \\ & + \frac{1}{\ell+1} \sum_{j=0}^{\ell} (-1)^{j+\ell}\, \overbrace{(\gamma\delta^{*}\gamma\delta^{*}\cdots)}^{(\ell-j) \text{ factors}}\, z_1\, \overbrace{(\cdots\gamma\delta^{*}\gamma\cdots)}^{j \text{ factors}} (a^{\dagger})^{\ell+1} \Bigg\},\end{aligned} \tag{121}$$

where $\ell = 0, 1, 2, \ldots$.

The superstates (119) and (120) can be written in the form of a supersqueeze operator acting on the supercoherent state, i.e.,

$$\begin{aligned}|\psi; -\rangle = {} & \mathcal{O}_{\text{even}}(a^{\dagger}, \gamma, \delta^{*}, z_1) \exp[-(\mathcal{O}_{\text{even}}(a^{\dagger}, \gamma, \delta^{*}, z_1))^{-1} \\ & \times (\mathcal{O}_{\text{odd}}(a^{\dagger}, \delta, \gamma^{*}, z_1))\, e^{2 z_1 a^{\dagger}} b^{\dagger}] \mathbb{D}(z_0) \mathbb{D}(z_1) |0; -\rangle \tilde{C}_0,\end{aligned} \tag{122}$$

$$\begin{aligned}|\psi; +\rangle = {} & \mathcal{O}_{\text{even}}(a^{\dagger}, \delta, \gamma^{*}, z_1) \exp[-(\mathcal{O}_{\text{even}}(a^{\dagger}, \delta, \gamma^{*}, z_1))^{-1} \\ & \times (\mathcal{O}_{\text{odd}}(a^{\dagger}, \gamma, \delta^{*}, z_1))\, e^{2 z_1 a^{\dagger}} b] \mathbb{D}(z_0) \mathbb{D}(z_1) |0; +\rangle \tilde{D}_0^{*},\end{aligned} \tag{123}$$

where

$$\mathcal{O}_{\text{even}}(a^{\dagger}, \gamma, \delta^{*}, z_1) = \sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{a^{\dagger}}(\ell, \gamma, \delta^{*}, z_1) \tag{124}$$

and

$$\mathcal{O}_{\text{odd}}(a^{\dagger}, \gamma, \delta^{*}, z_1) = \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{a^{\dagger}}(\ell, \gamma, \delta^{*}, z_1). \tag{125}$$

4.2.1. Standard Supersqueezed States

In the case where γ and δ are odd Grassmann numbers, i.e., when $\gamma = \gamma_1$ and $\delta = \delta_1$, it is easy to see from (121) that, the nonzero $\mathcal{O}_{a^{\dagger}}$ operators in (119) and (120) corresponds to

$$\mathcal{O}_{a^\dagger}(0,\gamma_1,-\delta_1,z_1)=1,\quad \mathcal{O}_{a^\dagger}(1,\delta_1,-\gamma_1,z_1)=\delta_1 a^\dagger-2\delta_1 z_1(a^\dagger)^2,$$
$$\mathcal{O}_{a^\dagger}(2,\gamma_1,-\delta_1,z_1)=-\frac{1}{2!}\gamma_1\delta_1(a^\dagger)^2, \tag{126}$$

and

$$\mathcal{O}_{a^\dagger}(0,\delta_1,-\gamma_1,z_1)=1,\quad \mathcal{O}_{a^\dagger}(1,\gamma_1,-\delta_1,z_1)=\gamma_1 a^\dagger-2\gamma_1 z_1(a^\dagger)^2,$$
$$\mathcal{O}_{a^\dagger}(2,\delta_1,-\gamma_1,z_1)=-\frac{1}{2!}\delta_1\gamma_1(a^\dagger)^2, \tag{127}$$

respectively. By inserting these results into (119) and (120), and after some simple manipulations, we get the supersqueezed states

$$|\psi;-\rangle=\exp\left[-\frac{1}{2}\gamma_1\delta_1(a^\dagger)^2\right]e^{-\delta_1 a^\dagger b^\dagger}e^{za^\dagger}|0;-\rangle C_0, \tag{128}$$

and

$$|\psi;+\rangle=\exp\left[-\frac{1}{2}\delta_1\gamma_1(a^\dagger)^2\right]e^{-\gamma_1 a^\dagger b}e^{za^\dagger}|0;+\rangle D_0^*, \tag{129}$$

which are eigenstates of $a+\gamma_1 b+\delta_1 b^\dagger$. In these last expressions, we notice the action of a normalizer operator acting on the corresponding supercoherent states. The normalizer in Eq. (128) transforms the algebra element $a+\gamma_1 b+\delta_1 b^\dagger$ into $a+\gamma_1 b$ whereas the normalizer in Eq. (129) transforms it into $a+\delta_1 b^\dagger$. In fact, a complete reduction into the element a only can be obtained. For instance, that is the case if we multiply the normalizer in Eq. (128) by the corresponding normalizer of Eq. (105) in the special case where $\gamma_0=0$, i.e., by $e^{-\gamma_1 a^\dagger b}$. Moreover, if we consider the algebra element $a+\beta_0 a^\dagger+\gamma_1 b+\delta_1 b^\dagger$, a normalizer operator transforming it into the element a is given by the standard supersqueeze operator (Buzano *et al.*, 1989)

$$\mathbf{G}(\beta_0,\gamma_1,\delta_1)=\exp\left[-(\beta_0+\gamma_1\delta_1)\frac{(a^\dagger)^2}{2}\right]\exp(-\delta_1 a^\dagger b^\dagger)\exp(-\gamma_1 a^\dagger b). \tag{130}$$

In this way, using the algebra eigenstates (117) of the a annihilator, we observe that a class of superalgebra eigenstates of $a+\beta_0 a^\dagger+\gamma_1 b+\delta_1 b^\dagger$, corresponding to the eigenvalue z_0, is given by

$$\mathbf{G}(\beta_0,\gamma_1,\delta_1)\mathbb{D}(z_0)\mathbb{T}(\theta_1)|0;-\rangle C_0. \tag{131}$$

We notice that, these supersequeezed states are obtained by acting with a supersqueeze operator that is an element of the osp(2/2) supergroup on the supercoherent states associated to the supersymmetric harmonic oscillator. In this way, these

SAES of the algebra element $a + \beta_0 a^\dagger + \gamma_1 b + \delta_1 b^\dagger$ are comparable to the supersqueezed states for the supersymmetric harmonic oscillator (Kostelecký *et al.*, 1993; Nieto, 1992).

4.2.2. Spin $\frac{1}{2}$ Representation AES Structure

Let us consider now the special case where both γ and δ are even invertible Grassmann numbers. Let us write $\gamma = \gamma_0$ and $\delta = \delta_0$. In this case, from (121), we obtain

$$\mathcal{O}_{a^\dagger}(\ell, \gamma_0, \delta_0, z_1) = \begin{cases} \dfrac{(a^\dagger)^\ell}{\ell!}(\gamma_0\delta_0)^{\ell/2} \exp\left(-\dfrac{\ell}{\ell+1} z_1 a^\dagger\right), & \text{if } \ell \text{ is even} \\ \dfrac{(a^\dagger)^\ell}{\ell!}(\gamma_0\delta_0)^{(\ell-1)/2}\gamma_0 \exp(-z_1 a^\dagger), & \text{if } \ell \text{ is odd} \end{cases}. \tag{132}$$

Thus, by inserting these results in (124) and (125), we get

$$\begin{aligned} \mathcal{O}_{\text{even}}(a^\dagger, \gamma_0, \delta_0, z_1) &= \sum_{\ell \text{ even}}^{\infty} \frac{(\sqrt{\gamma_0\delta_0}\, a^\dagger)^\ell}{\ell!} \exp\left(-\frac{\ell}{\ell+1} z_1 a^\dagger\right) \\ &= \cosh(\sqrt{\gamma_0\delta_0} a^\dagger)\, e^{-z_1 a^\dagger} \exp[z_1(\sqrt{\gamma_0\delta_0})^{-1} \\ &\quad \times (\cosh(\sqrt{\gamma_0\delta_0}\, a^\dagger))^{-1} \sinh(\sqrt{\gamma_0\delta_0}\, a^\dagger)] \end{aligned} \tag{133}$$

and

$$\begin{aligned} \mathcal{O}_{\text{odd}}(a^\dagger, \gamma_0, \delta_0, z_1) &= (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0} \sum_{\ell \text{ odd}}^{\infty} \frac{(\sqrt{\gamma_0\delta_0}\, a^\dagger)^\ell}{\ell!} \exp(-z_1\, a^\dagger) \\ &= (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0} \sinh(\sqrt{\gamma_0\delta_0}\, a^\dagger) \exp(-z_1 a^\dagger). \end{aligned} \tag{134}$$

By inserting these results into (119) and (120) and after some manipulations, we get the set of independent eigenstates of $a + \gamma_0 b + \delta_0 b^\dagger$:

$$\begin{aligned} |\psi; -\rangle &= \exp[-z_1(a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger))] \cosh\{\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1} \\ &\quad \times \sqrt{\delta_0}[1 + z_1(2a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger))] b^\dagger\}\, e^{z a^\dagger} |0; -\rangle C_0 \end{aligned} \tag{135}$$

and

$$\begin{aligned} |\psi; +\rangle &= \exp[-z_1(a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger))] \cosh\{\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1} \\ &\quad \times \sqrt{\gamma_0}[1 + z_1(2a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger))] b\}\, e^{z a^\dagger} |0; +\rangle D_0^*, \end{aligned} \tag{136}$$

where

$$T_h(\gamma_0, \delta_0, a^\dagger) = (\cosh(\sqrt{\gamma_0\delta_0} a^\dagger))^{-1} \sinh(\sqrt{\gamma_0\delta_0} a^\dagger). \tag{137}$$

In the special case where $z_1 = 0$, (135) and (136) reduce to

$$|\psi; -\rangle = \cosh[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0} b^\dagger]\, e^{z_0 a^\dagger}|0; -\rangle C_0$$
$$= -(\sqrt{\gamma}_0)^{-1}\sqrt{\delta_0}\sinh[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0} b]\, e^{z_0 a^\dagger}|0; +\rangle C_0 \quad (138)$$

and

$$|\psi; +\rangle = \cosh[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0} b]\, e^{z_0 a^\dagger}|0; +\rangle D_0^*$$
$$= -(\sqrt{\delta}_0)^{-1}\sqrt{\gamma_0}\sinh[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0} b^\dagger]\, e^{z_0 a^\dagger}|0; -\rangle D_0^*, \quad (139)$$

respectively. By combining both Eqs. (138) and (139), we can express the set of independent solutions in the form

$$|\widetilde{\psi; -}\rangle = \exp(\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0} b^\dagger)\, e^{z_0 a^\dagger}|0; -\rangle \tilde{C}_0 \quad (140)$$

and

$$|\widetilde{\psi; +}\rangle = \exp(\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0} b)\, e^{z_0 a^\dagger}|0; +\rangle \tilde{D}_0. \quad (141)$$

Thus, we recover the structure of the spin $\frac{1}{2}$ representation algebra eigenstates associated to the subalgebra $\{a, J_+, J_-\}$ of the $h(2) \otimes \mathrm{su}(2)$ Lie algebra (Alvarez-Moraga and Hussin, 2002).

4.3. The General Case

Let us solve now the eigenvalue Eq. (95). The discussion at the end of section 4.2.1 shows that it can be reduced to a simpler one by expressing the eigenstate $|\psi\rangle$ as

$$|\psi\rangle = \mathbf{G}(\beta_0, \gamma_1, \delta_1)|\varphi\rangle. \quad (142)$$

Indeed, inserting (142) into (95) and multiplying by the inverse of the supersqueeze operator $\mathbf{G}(\beta_0, \gamma_1, \delta_1)$, we get

$$[a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger]|\varphi\rangle = z|\varphi\rangle, \quad (143)$$

where

$$\hat{\beta}_1 = \beta_1 + \delta_0\gamma_1 + \gamma_0\delta_1 \in \mathbb{C}B_{L_1}. \quad (144)$$

We can show that (see Appendix B, section SAES of $a + \hat{\beta} a^\dagger + \gamma_0 b + \gamma_0 b^\dagger$) two classes of independent solutions of the eigenvalue Eq. (143) exit and are given by

$$|\varphi;-\rangle = \left[\sum_{\ell\,\text{even}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,\ell\right) \mathcal{O}_{a^\dagger}(\ell,\gamma_0,\delta_0,z_1)\, e^{za^\dagger}|0;-\rangle \right.$$
$$\left. - \sum_{\ell\,\text{odd}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,(\ell-1)\right) \mathcal{O}_{a^\dagger}(\ell,\delta_0,\gamma_0,z_1)\, e^{za^\dagger}|0;+\rangle \right] C_0 \tag{145}$$

and

$$|\varphi,+\rangle = \left[\sum_{\ell\,\text{even}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,\ell\right) \mathcal{O}_{a^\dagger}(\ell,\delta_0,\gamma_0,z_1)\, e^{za^\dagger}|0;+\rangle \right.$$
$$\left. - \sum_{\ell\,\text{odd}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,(\ell-1)\right) \mathcal{O}_{a^\dagger}(\ell,\gamma_0,\delta_0,z_1)\, e^{za^\dagger}|0;-\rangle \right] D_0^*, \tag{146}$$

where C_0 and D_0^* are arbitrary and invertible Grassmann constants.

Using the results (132) for the $\mathcal{O}_{a^\dagger}(\ell,\gamma_0,\delta_0,z_1)$ operator, we get

$$|\varphi;-\rangle = [\cosh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)(1+T_h(\gamma_0,\delta_0,\hat{\beta}_1,a^\dagger)\sqrt{\gamma_0\delta_0-\hat{\beta}_1 z_1})\,e^{-z_1a^\dagger}\,e^{za^\dagger}$$
$$\times|0;-\rangle - (\gamma_0)^{-1}\sinh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\sqrt{\gamma_0\delta_0+\hat{\beta}_1}\,e^{-z_1a^\dagger}\,e^{za^\dagger}|0;+\rangle]C_0 \tag{147}$$

and

$$|\varphi;+\rangle = [\cosh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}a^\dagger)(1+T_h(\gamma_0,\delta_0,\hat{\beta}_1,a^\dagger)\sqrt{\gamma_0\delta_0-\hat{\beta}_1 z_1})\,e^{-z_1a^\dagger}\,e^{za^\dagger}$$
$$\times|0;+\rangle - (\delta_0)^{-1}\sinh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\sqrt{\gamma_0\delta_0+\hat{\beta}_1}\,e^{-z_1a^\dagger}\,e^{za^\dagger}|0;-\rangle]D_0^*, \tag{148}$$

where

$$T_h(\gamma_0,\delta_0,\hat{\beta}_1,a^\dagger) = (\cosh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger))^{-1}\sinh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger). \tag{149}$$

4.3.1. Generalized Spin $\frac{1}{2}$ Representation AES Structure

In the special case where $z_1 = 0$, (147) and (148) reduce to

$$|\varphi;-\rangle = \exp\left(-\frac{1}{2}(\gamma_0)^{-1}\hat{\beta}_1 a^\dagger b^\dagger\right)$$
$$\cosh\left[\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger - (\gamma_0)^{-1}\sqrt{\gamma_0\delta_0+\hat{\beta}_1}\,b^\dagger\right] e^{z_0a^\dagger}|0;-\rangle C_0 \tag{150}$$

and

$$|\varphi;+\rangle = \exp\left(-\frac{1}{2}(\delta_0)^{-1}\hat{\beta}_1 a^\dagger b\right)$$
$$\cosh[\sqrt{\gamma_0\delta_0 - \hat{\beta}_1}\, a^\dagger - (\delta_0)^{-1}\sqrt{\gamma_0\delta_0 + \hat{\beta}_1}\, b]\, e^{z_0 a^\dagger}|0;+\rangle D_0^*, \quad (151)$$

respectively. Thus, we get a set of generalized SAES that contains the set of AES associated to the spin $\frac{1}{2}$ representation that we have studied in section 4.2.2.

5. ISOSPECTRAL HARMONIC OSCILLATOR HAMILTONIANS HAVING ODD INTERACTION TERMS

In this section we search for some isospectral harmonic oscillator systems which are characterized by a Hamiltonian admitting an annihilation operator which is a Grassmannian linear combination of the generators of the H–W Lie superalgebra, i.e., of the form

$$\mathcal{A} = a + \beta a^\dagger + \gamma b + \delta b^+, \qquad \beta, \gamma, \delta, \in \mathbb{C}B_L. \quad (152)$$

A family of nonequivalent such Hamiltonians $\mathcal{H}$ can be constructed if first we consider a super-Hermitian Hamiltonian $\mathcal{H}_0$ such that the commutator is given by

$$[\mathcal{H}_0, \mathcal{A}_0] = -\mathcal{A}_0 \quad \text{and} \quad \mathcal{A}_0|E_0;\pm\rangle = 0, \quad (153)$$

where

$$\mathcal{A}_0 = a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger, \qquad \gamma_0, \delta_0 \in \mathbb{C}B_{L_0}, \quad (154)$$

$\hat{\beta}_1$ is given by (144) and $|E_0;\pm\rangle$ are the zero eigenvalue eigenstates of $\mathcal{H}_0$. In this way, $\mathcal{A}_0$ is effectively an annihilation operator and its associated superalgebra eigenstates a class of supercoherent states for the system characterized by the Hamiltonian $\mathcal{H}_0$. Second, according to the analysis of Appendix B, section SAES of $a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger$, it is possible to construct $\mathcal{H}$ satisfying

$$[\mathcal{H}, \mathcal{A}] = -\mathcal{A} \quad (155)$$

by taking

$$\mathcal{A} = \boldsymbol{G}(\beta_0, \gamma_1, \delta_1)\mathcal{A}_0(\boldsymbol{G}(\beta_0, \gamma_1, \delta_1))^{-1} \quad \text{and}$$
$$\mathcal{H} = \boldsymbol{G}(\beta_0, \gamma_1, \delta_1)\mathcal{H}_0(G(\beta_0, \gamma_1, \delta_1))^{-1}, \quad (156)$$

where $\boldsymbol{G}(\beta_0, \gamma_1, \delta_1)$ is the standard supersqueeze operator defined in (130). We see that our original problem thus reduces to one of finding $\mathcal{H}_0$. We observe that the Hamiltonian $\mathcal{H}$ in (156) is not super-Hermitian but it belongs to a class of Hamiltonians that generalize the one of η-pseudo-Hermitian Hamiltonians (Mostafazadeh, 2002). Indeed, it satisfies the relation

$$\mathcal{H}^\ddagger = \eta\mathcal{H}\eta^{-1}, \quad (157)$$

where η is the super-Hermitian operator

$$\eta = (\boldsymbol{G}^{-1}(\beta_0, \gamma_1, \delta_1))^{\ddagger} \boldsymbol{G}^{-1}(\beta_0, \gamma_1, \delta_1). \tag{158}$$

Let us mention that a family of $\mathcal{H}_0$-equivalent Hamiltonians can be obtained if we replace $\boldsymbol{G}(\beta_0, \gamma_1, \delta_1)$ in (156) by a suitable osp(2/2) superunitary operator (Buzano *et al.*, 1989)

$$\boldsymbol{U}(\chi_0, \Gamma_1, \Delta_1) = \exp\left(\chi_0 \frac{(a^\dagger)^2}{2} - \chi_0^{\ddagger} \frac{a^2}{2} + \Gamma_1 a^\dagger b^\dagger + \Gamma_1^{\ddagger} ab + \Delta_1 a^\dagger b + \Delta_1^{\dagger} ab^\dagger\right), \tag{159}$$

where $\chi_0 \in \mathbb{C}B_{L_0}$ and $\Gamma_1, \Delta_1, \in \mathbb{C}B_{L_1}$.

Let us also mention that if we denote $\mathcal{A}_0^{\ddagger}$ the adjoint of $\mathcal{A}_0$, then the usual commutator leads to

$$\begin{aligned}[\mathcal{A}_0, \mathcal{A}_0^{\ddagger}] &= 1 - \hat{\beta}_1^{\ddagger} \hat{\beta}_1 \{a, a^\dagger\} + (\delta_0^{\ddagger} \delta_0 - \gamma_0^{\ddagger} \gamma_0)[b^\dagger, b] \\ &\quad + 2\hat{\beta}_1 \delta_0^{\ddagger} a^\dagger b - 2\delta_0 \hat{\beta}_1^{\ddagger} ab^\dagger + 2\hat{\beta}_1 \gamma_0^{\ddagger} a^\dagger b^\dagger - 2\gamma_0 \hat{\beta}_1^{\ddagger} ab, \end{aligned} \tag{160}$$

and we notice that, under the conditions $\gamma_0 = \delta_0 = 0$, or $\hat{\beta}_1 = 0$, the commutator (160) becomes a diagonal operator in the Fock vector basis $\{|n, \pm\rangle, n \in \mathbb{N}\}$.

5.1. *h*(2) Generalized Isospectral Oscillator System

Let us here consider the particular case where $\gamma_0 = \delta_0 = 0$. In this case, the operator $\mathcal{A}_0$ takes the simple form

$$\mathcal{A}_0 = a + \hat{\beta}_1 a^\dagger \tag{161}$$

and the commutator (160) writes

$$[\mathcal{A}_0, \mathcal{A}_0^{\ddagger}] = 1 - \hat{\beta}_1^{\ddagger} \hat{\beta}_1 \{a, a^\dagger\}. \tag{162}$$

A class of Hamiltonian $\mathcal{H}_0$ satisfying (153) is given by

$$\begin{aligned}\mathcal{H}_0 &= (1 + \hat{\beta}_1^{\ddagger} \hat{\beta}_1)[\mathcal{A}_0^{\dagger} \mathcal{A}_0 + \hat{\beta}_1^{\ddagger} \hat{\beta}_1 (a^\dagger)^2 a^2] \\ &= a^\dagger a + \hat{\beta}_1 (a^\dagger)^2 + \hat{\beta}_1^{\ddagger} a^2 + \hat{\beta}_1^{\ddagger} \hat{\beta}_1 (a^\dagger a + aa^\dagger) + \hat{\beta}_1^{\ddagger} \hat{\beta}_1 (a^\dagger)^2 a^2. \end{aligned} \tag{163}$$

We notice that we are in presence of a super-Hermitian Hamiltonian of the harmonic oscillator type with nilpotent interaction terms which contain odd contributions. We also notice that this hamiltonian can be expressed in the form

$$\mathcal{H}_0 = \frac{\mathcal{N}}{2} + \mathcal{M} + \mathcal{Q}_+ + \mathcal{Q}_-, \tag{164}$$

where

$$\mathcal{N} = 2\hat{\beta}_1^{\ddagger} \hat{\beta}_1 (a^\dagger a + aa^\dagger), \quad \mathcal{Q}_+ = \hat{\beta}_1 (a^\dagger)^2, \quad \mathcal{Q}_- = \hat{\beta}_1^{\ddagger} a^2, \quad \mathcal{M} = a^\dagger a - \mathcal{Q}_+ \mathcal{Q}_-. \tag{165}$$

The nonzero supercommutation relations between these operators are given by

$$[\mathcal{M}, \mathcal{Q}_\pm] = \pm 2\mathcal{Q}_\pm, \qquad \{\mathcal{Q}_+, \mathcal{Q}_-\} = \mathcal{N}, \tag{166}$$

i.e., they have almost the structure of $u(1/1)$ superalgebra. Indeed, here $\mathcal{N}$ is an even nilpotent operator such that $\mathcal{N}^2 = 0$.

According to (153) and (163), a class of superalgebra eigenstates of $\mathcal{H}_0$ can be obtained by applying n times ($n = 0, 1, 2, \ldots$) the raising operator $\mathcal{A}_0^\dagger$ on the zero eigenvalue eigenstates of $\mathcal{A}_0$. From (45), we deduce that these latter are given by

$$|E_0; j\rangle = \left(1 - \frac{1}{4}\hat{\beta}_1^\ddagger \hat{\beta}_1\right)\left[|0; j\rangle - \frac{\hat{\beta}_1}{\sqrt{2}}|2; j\rangle\right], \tag{167}$$

where j corresponds to the set $\{-, +\}$.

Then, as $\mathcal{H}_0|E_0; j\rangle = 0$, the generated energy eigenstates are given by

$$|E_n; j\rangle \propto (A_0^\ddagger)^n |E_0; j\rangle = \left((a^\dagger)^n + \hat{\beta}_1^\ddagger \sum_{k=0}^{n-1} (a^\dagger)^{(n-1-k)} a (a^\dagger)^k\right)|E_0; j\rangle \tag{168}$$

and the corresponding energy eigenvalues are $E_n^j = n$. An orthonormalized version of these states is given by

$$|E_n; j\rangle = \left(1 - \frac{1}{4}\hat{\beta}_1^\ddagger \hat{\beta}_1 (2n+1)\right)$$
$$\left[|n; j\rangle + \frac{\hat{\beta}_1^\ddagger}{2}\sqrt{n(n-1)}|n-2; j\rangle - \frac{\hat{\beta}_1}{2}\sqrt{(n+1)(n+2)}|n+2; j\rangle\right], \tag{169}$$

where $n \in \mathbb{N}$. From (169), it is easy to calculate the action of $\mathcal{A}_0^\ddagger$ and $\mathcal{A}_0$ on the $|E_n; j\rangle$ eigenstates, we get

$$\mathcal{A}_0^\ddagger |E_n; j\rangle = \left(1 - \frac{1}{2}\hat{\beta}_1^\ddagger \hat{\beta}_1 (n+1)\right)\sqrt{n+1}|E_{n+1}; j\rangle \tag{170}$$

and

$$\mathcal{A}_0 |E_n; j\rangle = \left(1 - \frac{1}{2}\hat{\beta}_1^\ddagger \hat{\beta}_1 n\right)\sqrt{n}|E_{n-1}; j\rangle. \tag{171}$$

Thus, the orthonormalized energy eigenstates $|E_n; j\rangle$ can be written in the standard form

$$|E_n; j\rangle = \left(1 + \frac{1}{4}\hat{\beta}_1^\ddagger \hat{\beta}_1 n(n+1)\right)\frac{(A_0^\ddagger)^n}{\sqrt{n!}}|E_0; j\rangle. \tag{172}$$

This is a complete set of states. Indeed, using (169), we can demonstrate the completeness property

$$\sum_j \sum_{n=0}^{\infty} |E_n; j\rangle\langle E_n; j| = I \otimes I = \sum_j \sum_{n=0}^{\infty} |n; j\rangle\langle n; j|. \tag{173}$$

On the other hand, we can express the $|n; j\rangle$ states in the form

$$|n; j\rangle = \left(1 - \frac{1}{4}\hat{\beta}_1^{\ddagger}\hat{\beta}_1(2n+1)\right)\left[|E_n; j\rangle - j\sqrt{(n+1)(n+2)}|E_{n+2}; j\rangle\frac{\hat{\beta}_1}{2}\right.$$
$$\left. + j\sqrt{n(n-1)}|E_{n-2}; j\rangle\frac{\hat{\beta}_1^{\ddagger}}{2}\right], \tag{174}$$

then, from (172) and after some manipulations, we get

$$|0; j\rangle = \left(1 - \frac{1}{4}\hat{\beta}_1^{\ddagger}\hat{\beta}_1\right)\exp\left(\frac{(\mathcal{A}_0^{\ddagger})^2}{2}\hat{\beta}_1\right)|E_0; j\rangle. \tag{175}$$

According to (45), the coherent states associated to a physical system characterized by the hamiltonian (163) can be written as

$$|\varphi; j\rangle = \exp\left[-\hat{\beta}_1\frac{(a^{\dagger})^2}{2} - \hat{z}_1\hat{\beta}_1\frac{(a^{\dagger})^3}{3}\right]\mathbb{D}(\hat{z}_0)\mathbb{D}(\hat{z}_1)$$
$$\times\left(1 - \frac{1}{4}\hat{\beta}_1^{\ddagger}\hat{\beta}_1\right)\exp\left(\frac{(\mathcal{A}_0^{\ddagger})^2}{2}\hat{\beta}_1\right)|E_0; j\rangle\hat{C}(\hat{z}, \hat{\beta}_1). \tag{176}$$

5.2. Spin $\frac{1}{2}$ Generalized Isospectral Oscillator System

In the case where $\hat{\beta}_1 = 0$ and $\gamma_0^{\ddagger}\gamma_0 = \delta_0^{\ddagger}\delta_0$, the operator $\mathcal{A}_0$ takes the form

$$\mathcal{A}_0 = a + \gamma_0 b + \delta_0 b^{\dagger} \tag{177}$$

and the commutator (160) writes

$$[\mathcal{A}_0, \mathcal{A}_0^{\ddagger}] = 1. \tag{178}$$

A class of Hamiltonian $\mathcal{H}_0$ satisfying (153) is given by

$$\mathcal{H}_0 = \mathcal{A}_0^{\ddagger}\mathcal{A}_0 = a^{\dagger}a + \gamma_0^{\ddagger}\gamma_0 + \gamma_0 a^{\dagger}b + \gamma_0^{\dagger}ab^{\dagger} + \delta_0 a^{\dagger}b^{\dagger} + \delta_0^{\dagger}ab. \tag{179}$$

We notice that this is a super-Hermitian Hamiltonian, without defined parity, which is a linear Grassmann combination of generators of the osp(2/2) Đ sh(2/2) Lie superalgebra. Then, in this aspect, the corresponding Hamiltonian $\mathcal{H}$ defined in (156) complements the classes of Hamiltonians considered by Buzano *et al.* (1989).

By construction, the eigenstates of $\mathcal{A}_0$ corresponding to the eigenvalue $z = 0$ are eigenstates of $\mathcal{H}_0$ corresponding to the eigenvalue $E_0 = 0$. Let us take these states to be the normalized version of states (140 and 141), when $z_0 = 0$, i.e.,

$$|E_0, -\rangle = \left(\sqrt{1 + (\sqrt{\gamma_0})^{-1}((\sqrt{\gamma_0})^{-1})^{\ddagger}\sqrt{\delta_0}(\sqrt{\delta_0})^{\ddagger}}\right)^{-1} \times \mathbb{D}(\sqrt{\gamma_0\delta_0})[|0; -\rangle - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0}|0; +\rangle] \tag{180}$$

and

$$|E_0, +\rangle = \left(\sqrt{1 + (\sqrt{\delta_0})^{-1}((\sqrt{\delta_0})^{-1})^{\ddagger}\sqrt{\gamma_0}(\sqrt{\gamma_0})^{\ddagger}}\right)^{-1} \times \mathbb{D}(\sqrt{\gamma_0\delta_0})[|0; +\rangle - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0}|0; -\rangle]. \tag{181}$$

Thus, from (153) and (178), we deduce that a class of orthonormalized eigenstates of $\mathcal{H}_0$ corresponding to the eigenvalue $E_n^j = n$ is given by ($n = 0, 1, 2, \ldots;$ $j = -, +$)

$$|E_n, j\rangle = \frac{(\mathcal{A}_0^{\ddagger})^n}{\sqrt{n!}}|E_0, j\rangle. \tag{182}$$

Moreover, a class of normalized coherent states for this generalized harmonic system which are eigenstates of $\mathcal{A}_0$ corresponding to the eigenvalue $z = z_0$ is easily constructed as (Alvarez-Moraga and Hussin, 2002)

$$|z_0, j\rangle = \exp(z_0\mathcal{A}_0^{\ddagger} - z_0^{\ddagger}\mathcal{A}_0)|E_0, j\rangle. \tag{183}$$

These coherent states are obtained from those of Eqs. (140 and 141) by acting with the following superunitary transformation:

$$\mathcal{U}(z_0; \gamma_0, \delta_0) = \exp\left[z_0\left(\gamma_0^{\ddagger}b^{\dagger} + \delta_0^{\ddagger}b\right) - z_0^{\ddagger}(\gamma_0 b + \delta_0 b^{\dagger})\right]. \tag{184}$$

6. CONCLUSIONS

In this paper we have generalized the AES (Brif, 1997) concept to the one of SAES. We have demonstrated that SAES associated to the H–W Lie superalgebra contain the sets of standard coherent and supercoherent states associated to the usual and supersymmetric harmonic oscillator systems, respectively (Alvarez-Moraga and Hussin, 2002; Aragone and Zypman, 1986; Fatyga *et al.*, 1991; Perelomov, 1986). Also, these SAES contain both the standard squeezed and supersqueezed states (Nieto, 1992; Orszag and Salamo, 1988) and the supersqueezed states associated to the spin$-\frac{1}{2}$ representation of the AES of the $h(2) \oplus su(2)$ algebra (Alvarez-Moraga and Hussin, 2002). Let us mention that the introduction of Grassmann coefficients in the linear combination of the superalgebra generators helps us to understand the role played by the c-numbers (even Grassmann numbers) and d-numbers (odd Grassmann numbers) interaction coefficients, in the mentioned literature. Moreover, from the idea of giving to SAES the interpretation of an operator associated to a physical system, we have constructed some classes of super-Hermitian and η-pseudo-super-Hermitian Hamiltonians (DeWitt, 1984; Mostafazadeh, 2002), isospectral to the standard harmonic oscillator hamiltonian. We have found their physical eigenstates and their associated supercoherent states. In this respect, we see that the SAES concept constitutes an alternative and unified approach for the construction of generalized coherent and supercoherent and also squeezed and supersqueezed states for a given quantum system.

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APPENDIX A

Notations and Conventions

In this Appendix we want to fix the notations and conventions used in this work. They concern principally the concepts of Grassmann algebra, Lie superalgebra and their representations, super-Hermitian and superunitary operators, super Lie algebra, and linear Lie supergroup.

Let us remind that a complex *Grassmann algebra*, $\mathbb{C}B_L$, is a linear vector space over the field of complex numbers, associative and Z_2 graded. It may thus be decomposed into $\mathbb{C}B_{L_0} + \mathbb{C}B_{L_1}$, where the even space $\mathbb{C}B_{L_0}$ is generated by the set of 2^{L-1} linearly independent generators $\mathcal{E}_\mu$ of even level and the odd space $\mathbb{C}B_{L_1}$ is generated by the set of 2^{L-1} linearly independent generators $\mathcal{E}_\mu$ of odd level. Here, the index μ represents either the empty set ϕ or the set $(j_1, j_2, \ldots, j_{N(\mu)})$ of $N(\mu)$ integer numbers such that $1 \leq j_1 < j_2 \cdots < j_{N(\mu)} \leq L$. $N(\mu)$ is the level of the generator $\mathcal{E}_\mu$. The identity of the algebra is $\mathcal{E}_\phi = \mathbf{1}$ and $\mathcal{E}_\mu = \mathcal{E}_{j1}\mathcal{E}_{j2}\cdots\mathcal{E}_{jN(\mu)}$ is the ordered product of $N(\mu)$ odd generators of level 1 taken among the set of basic generators $\{\mathcal{E}_j,\ j = 1, 2, \ldots, L\}$. The product of these generators is associative and antisymmetric. Moreover, any nonzero product of the type $\mathcal{E}_{j1}\mathcal{E}_{j2}\ldots\mathcal{E}_{jr}$ of r generators is linearly independent of the products containing less than r generators and we have $\mathcal{E}_\phi\mathcal{E}_j = \mathcal{E}_j\mathcal{E}_\phi = \mathcal{E}_j, \forall_j = 1, 2, \ldots, L$. The graduation is introduced by defining the degree of $\mathcal{E}_\mu$, i.e.,

$$\deg \mathcal{E}_\mu = (-1)^{N(\mu)}, \tag{185}$$

with $N(\phi) = 0$.

Any element $B \in \mathbb{C}B_L$ can be written either in the form

$$B = \sum_{\mu} B_\mu \mathcal{E}_\mu, \qquad B_\mu \in \mathbb{C}, \tag{186}$$

or as the sum of its even part B_0 and its odd part B_1, i.e., $B = B_0 + B_1$ with

$$B_0 = \sum_{\text{even } N(\mu)} B_\mu \mathcal{E}_\mu, \qquad B_1 = \sum_{\text{odd } N(\mu)} B_\mu \mathcal{E}_\mu. \tag{187}$$

We also deduce the graded operations for the Grassmman algebra, i.e., for all $B_0, Z_0 \in \mathbb{C}B_{L_0}$, $B_1, Z_1 \in \mathbb{C}B_{L_1}$, we have

$$B_0 Z_0 = Z_0 B_0 \in \mathbb{C}B_{L_0},\ B_0 Z_1 = Z_1 B_0 \in \mathbb{C}B_{L_1},\ B_1 Z_1 = -Z_1 B_1 \in \mathbb{C}B_{L_0}. \tag{188}$$

In particular, for all $B = B_0 + B_1 \in \mathbb{C}B_L$ and $Z_1 \in \mathbb{C}B_{L_1}$,

$$BZ_1 = Z_1 B^*, \qquad Z_1 B = B^* Z_1, \tag{189}$$

where

$$B^* = B_0 - B_1, \tag{190}$$

is the *conjugate* of B. The product of any two elements of the algebra, B and B', corresponds to

$$BB' = \sum_{\mu} \sum_{\mu'} B_\mu B'_{\mu'} (\mathcal{E}_\mu \mathcal{E}_{\mu'}), \tag{191}$$

with

$$\mathcal{E}_\mu \mathcal{E}_{\mu'} = \pm \mathcal{E}_\nu, \quad \text{where} \quad N(\nu) = N(\mu) + N(\mu'), \tag{192}$$

when neither of the indices in the sets represented by μ and μ' is repeated, and $\mathcal{E}_\mu \mathcal{E}_{\mu'} = 0$, when at least one of the index in the set represented by μ and μ' is repeated. The sign $\pm$ in (192) is determined by using the antisymmetric property of the basic generators $\mathcal{E}_j$ when reordering the their product.

The identity component of the element B, usually called the body, is denoted by $\epsilon(B) = B_\phi \in \mathbb{C}$, whereas the nilpotent quantity $s(B) = B - B_\phi \mathcal{E}_\phi$ defines the soul of B.

With respect to the *complex conjugate* of the element $B \in \mathbb{C}B_L$, we follow the conventions of Cornwell (1989) and thus write

$$\overline{B} = \sum_{\mu} \overline{B}_\mu \mathcal{E}_\mu, \tag{193}$$

i.e., the basis elements $\mathcal{E}_\mu$ are considered as the real Grassmann numbers. Also, the *adjoint* of B is defined by the relation

$$B^\ddagger = \sum_{\mu} \overline{B}_\mu \mathcal{E}_\mu^\ddagger, \tag{194}$$

where

$$\mathcal{E}_\mu^\ddagger = \begin{cases} \mathcal{E}_\mu, & \text{if } N(\mu) \text{ is even} \\ -i\mathcal{E}_\mu, & \text{if } N(\mu) \text{ is odd.} \end{cases} \tag{195}$$

This adjoint operation has the same properties than the ones of the usual adjoint operation for complex matrices.

The inverse of a Grassmann number B, denoted by $(B)^{-1}$, is defined as

$$B(B)^{-1} = (B)^{-1}B = \epsilon_\phi = 1. \tag{196}$$

It is important to mention that B is invertible if and only if $B_\phi \neq 0$.

The integration with respect to an odd Grassmann variable must be considered in the Berezin sense (Berezin, 1987), i.e., if $\eta \in \mathbb{C}B_{L_1}$, then

$$\int d\eta = 0, \qquad \int \eta\, d\eta = 1, \tag{197}$$

where the integration is taken over all the domain of definition of η.

Let us now recall some useful definitions and properties of Lie superalgebras, supergroups, and associated representations.

Definition A1. A (m/n) dimensional complex Lie superalgebra $\mathcal{L}_s$, is a complex vector space, Z_2 graded with respect to a generalized Lie product, formed from the direct sum of two subspaces, the even subspace of dimension $m \geq 0$, which we denote by $\mathcal{L}_0$, and the odd subspace of dimension $n \geq 0\,(m + n \geq 1)$, which we denote by $\mathcal{L}_1$, such that, for all $a, b \in \mathcal{L}_s$, there exists a generalized Lie product (supercommutator) $[a, b]$ with the following properties:

1) $[a, b] \in \mathcal{L}_s$, for all $a, b \in \mathcal{L}_s$;
2) for all $a, b, c \in \mathcal{L}_s$ and any complex (real) numbers α and β,

$$[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]; \tag{198}$$

3) if a and b are homogeneous elements of $\mathcal{L}_s$ then $[a, b]$ is also a homogeneous element of $\mathcal{L}_s$ whose degree is $(\deg a + \deg b)$ mod 2; that is, $[a, b]$ is odd if either a or b is odd, but $[a, b]$ is even if a and b are both even or if a and b are both odd;
4) for any homogeneous elements a and b of $\mathcal{L}_s$

$$[b, a] = -(-1)^{(\text{dega})(\text{debb})}[a, b]; \quad \text{and} \tag{199}$$

5) for any three homogeneous elements a, b, and c of $\mathcal{L}_s$, we have the generalized Jacobi identity:

$$[a, [b, c]](-1)^{(\deg a)(\deg c)} + [b, [c, a]](-1)^{(\deg b)(\deg a)} + [a, [b, c]](-1)^{(\deg c)(\deg b)} = 0. \tag{200}$$

We notice that the even subspace, $\mathcal{L}_0$, is an ordinary complex Lie algebra whereas the odd subspace, $\mathcal{L}_1$, is a carrier space for a representation of a Lie algebra $\mathcal{L}_0$.

Just as an ordinary Lie algebra can, in general, be represented by a set of complex matrices, a Lie superalgebra can also be represented, in general, by a set of complex matrices. Nevertheless, the graded character of a superalgebra implies certain special conditions for the structure of these matrices.

Definition A2. Suppose that for every $a \in \mathcal{L}_s$, there exists a matrix $\Gamma(a)$ from the set of complex matrices partitioned in the form $(d_0/d_1) \times (d_0/d_1)$, that we denote by $M(d_0/d_1; \mathbb{C})$, such that

1) for all $a, b \in \mathcal{L}_s$ and α, β of the field of $\mathcal{L}_s$,

$$\Gamma(\alpha a + \beta b) = \alpha\Gamma(a) + \beta\Gamma(b); \tag{201}$$

2) for all $a, b \in \mathcal{L}_s$,

$$\Gamma([a, b]) = [\Gamma(a), \Gamma(b)]; \tag{202}$$

3) if $a \in \mathcal{L}_0$, the even subspace of $\mathcal{L}_s$, then $\Gamma(a)$ a la forme

$$\Gamma(a) = \begin{pmatrix} \Gamma_{00}(a) & \mathbf{0} \\ \mathbf{0} & \Gamma_{11}(a) \end{pmatrix}, \tag{203}$$

where $\Gamma_{00}(a)$ and $\Gamma_{11}(a)$ are $d_0 \times d_0$ and $d_1 \times d_1$ dimensional submatrices, respectively; and if $a \in \mathcal{L}_1$, the odd subspace of $\mathcal{L}_s$, then $\Gamma(a)$ has the form

$$\Gamma(a) = \begin{pmatrix} \mathbf{0} & \Gamma_{01}(a) \\ \Gamma_{10}(a) & \mathbf{0} \end{pmatrix}, \tag{204}$$

where $\Gamma_{01}(a)$ and $\Gamma_{10}(a)$ are $d_0 \times d_1$ and $d_1 \times d_0$ dimensional submatrices, respectively. Then these matrices $\Gamma(a)$ are said to form a (d_0/d_1)-dimensional *graded representation* of $\mathcal{L}_s$.

Let $\mathcal{L}_s$ be a (m/n) dimensional complex Lie superalgebra with even basis elements $a_1, a_2, \ldots, a_m$ and odd basis elements $a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, represented by the set of matrices $\Gamma(a_k)$, $k = 1, 2, \ldots, m + n$. To each matrix $\Phi(a_k)$, we can associate a linear operator $\Phi(a_k)$ acting on the carrier space $\mathcal{W}$ of the representation. This space is a $(d_0 + d_1)$ inner product vector space expanded by a basis formed by the set of even vectors $\{|w_j\rangle\}_{j=0}^{d_0}$ and the set of odd vectors $\{|w_j\rangle\}_{j=d_0+1}^{d_0+d_1}$ and this action is defined by the relation

$$\Phi(a_k)|w_j\rangle = \sum_{i=1}^{d_0+d_1} (\Gamma(a_k))_{ij}|w_i\rangle. \tag{205}$$

Then $\mathcal{L}_s$ can also be represented by set of even operators $\Phi(a_k)(k = 1, 2, \ldots, m)$ and the set of odd opertors $\Phi(a_k)(k = m+1, m+2, \ldots, m+n)$, verifying the same supercommutation relations as the basis elements $a_k(k = 1, 2, \ldots, m+n)$.

Let χ to be a polynomial function of the $\mathcal{L}_s$ superalgebra generators, with complex Grassmannian coefficients. We say that χ is a *super-Hermitian* (anti-super-Hermitian) operator if $\chi = \chi^{\ddagger}(\chi = -\chi^{\ddagger})$. In particular, if χ is a complex Grassmannian linear combination of the $\mathcal{L}_s$ superalgebra generators, i.e.,

$$\chi = \sum_{j=1}^{m} C^j \Phi(a_j) + \sum_{k=1}^{n} D^k \Phi(a_{m+k}), \tag{206}$$

where $C^j \in \mathbb{C}B_L(j = 1, 2 \ldots, m)$ and $D^k \in \mathbb{C}B_L(k = 1, 2 \ldots, n)$ then

$$\chi^{\ddagger} = \sum_{j=1}^{m} (\Phi(a_j))^{\dagger}(C^j)^{\ddagger} + \sum_{k=1}^{n} (\Phi(a_{m+k}))^{\dagger}(D^k)^{\ddagger}, \tag{207}$$

where the † symbol is reserved for the usual adjoint operation. We say that a general $\mathcal{U}$ operator is *superunitary* if $\mathcal{U}\mathcal{U}^{\ddagger} = \mathcal{U}^{\ddagger}\mathcal{U} = I$, where I, is the identity operator. In particular, if χ is an anti-super-Hermitian operator, then $\mathcal{U} = e^{\chi}$ is a superunitary operator.

If for $j = 1, 2, \ldots, m$ and every element $\mathcal{E}_\mu$ of $\mathbb{C}B_L$, we define the even operators

$$M_\mu^j = \mathcal{E}_\mu \Phi(a_j) \tag{208}$$

and for $k = 1, 2, \ldots, n$ and every odd element $\mathcal{E}_\nu$ of $\mathbb{C}B_L$, we define the even operators

$$N_\nu^k = \mathcal{E}_\nu \Phi(a_{m+k}), \tag{209}$$

then the set of $(m+n)2^{L-1}$ operators defined by Eqs. (208) and (209) form a basis of a $(m+n)2^{L-1}$ dimensional real Lie algebra, whose Lie product is given by the usual commutator induced by the generalized Lie product of $\mathcal{L}_s$. This real Lie algebra is denoted by $\mathcal{L}_s(\mathbb{C}B_L)$ and is called a *super Lie algebra*. A general element M of this super Lie algebra writes

$$M = \sum_{j=1}^{m} \sum_{\text{even}\,\mu} X_\mu^j M_\mu^j + \sum_{k=1}^{n} \sum_{\text{odd}\,\nu} \Theta_\nu^k N_\nu^k, \tag{210}$$

where X_μ^j and Θ_ν^k are real parameters. Also we can write this element in the form

$$M = \sum_{j=1}^{m} X^j M^j + \sum_{k=1}^{n} \Theta^k N^k, \tag{211}$$

where $X^j = \sum_{\text{even}\,\mu} X^j_\mu \mathcal{E}_\mu \in \mathbb{R}B_{L_0}$, $\Theta^k = \sum_{\text{odd}\,\nu} \Theta^k_\nu \mathcal{E}_\nu \in \mathbb{R}B_{L_1}$ and

$$M^j = \mathcal{E}_\phi \Phi(a_j), \quad N^k = \mathcal{E}_\phi \Phi(a_{m+k}). \tag{212}$$

Let us end this Appendix by giving a method of construction of a linear Lie *supergroup* (Rogers, 1981). If $\mathcal{L}_s(\mathbb{C}B_L)$ is a real super Lie algebra whose basis elements are defined by (208) and (209), then every linear Lie group whose associated real Lie super algebra is given by $\mathcal{L}_s(\mathbb{C}B_L)$ is a (m/n) linear Lie supergroup, which we denote by $\mathcal{G}_s(\mathbb{C}B_L)$. The elements near the identity can be parametrized by

$$\mathbf{G}(\mathbf{X}; \boldsymbol{\Theta}) = \exp\{M\} = \exp\left\{\sum_{j=1}^{m} X^j M^j + \sum_{k=1}^{n} \Theta^k N^k\right\}. \tag{213}$$

APPENDIX B

Solving $[\boldsymbol{a} + \beta \boldsymbol{a}^\dagger + \gamma \boldsymbol{b} + \delta \boldsymbol{b}^\dagger]|\psi\rangle = \boldsymbol{z}|\psi\rangle$

In this Appendix we will solve the eigenvalue Eq. (95). We will do it in two steps. Firstly, we will solve the eigenvalue Eq. (118) and express its solutions in terms of a generalized supersqueeze operator acting on the supercoherent states $e^z|0;\pm\rangle$. This supersqueeze operator is used to reduce the eigenvalue Eq. (95) to a simpler one (see section 4.3) that is to the eigenvalue equation (143). Finally, we will solve the eigenvalue Eq. (143).

SAES of $a + \gamma b + \delta b^\dagger$

Let us solve the eigenvalue Eq. (118). The solution is assumed on the type (97) and by inserting it into (118), then using the usual properties of the operators and the states $\{|n;\pm\rangle\}$, we get the system $(n = 0, 1, 2\ldots)$

$$\sqrt{n+1}C_{n+1} + \gamma D_n^* = zCn, \tag{214}$$

$$\sqrt{n+1}D_{n+1} + \delta C_n^* = zDn. \tag{215}$$

Let us notice the symmetric form of this system. Proceeding by iteration, we can express the C_n and D_n coefficients in terms of the arbitrary Grassmann constants C_0 and D_0, i.e. $(n = 1, 2, \ldots)$,

$$C_n = \frac{1}{\sqrt{n}!}\left\{z^n C_0 - \sum_{k_1=0}^{(n-1)} z^{(n-1-k_1)}\gamma(z^*)^{k_1} D_0^* \right.$$
$$+ \sum_{k_1=0}^{(n-2)}\sum_{k_2=0}^{(n-2-k_1)} z^{(n-2-k_1-k_2)}\gamma(z^*)^{k_2}\delta^* z^{k_1} C_0$$

$$- \sum_{k_1=0}^{(n-3)} \sum_{k_2=0}^{(n-3-k_1)} \sum_{k_3=0}^{(n-3-k_1-k_2)} z^{(n-3-k_1-k_2-k_3)} \gamma (z^*)^{k_3} \delta^* z^{k_2} \gamma (z^*)^{k_1} D_0^* + \cdots$$

$$+ (-1)^n (\gamma \delta^*)^{[\frac{n}{2}]} \gamma^{(n-2[\frac{n}{2}])} F_{n-2[\frac{n}{2}]} \Bigg\}, \tag{216}$$

and

$$D_n = \frac{1}{\sqrt{n!}} \Bigg\{ z^n D_0 - \sum_{k_1=0}^{(n-1)} z^{(n-1-k_1)} \delta (z^*)^{k_1} C_0^*$$

$$+ \sum_{k_1=0}^{(n-2)} \sum_{k_2=0}^{(n-2-k_1)} z^{(n-2-k_1-k_2)} \delta (z^*)^{k_2} \gamma^* z^{k_1} D_0$$

$$- \sum_{k_1=0}^{(n-3)} \sum_{k_2=0}^{(n-3-k_1)} \sum_{k_3=0}^{(n-3-k_1-k_2)} z^{(n-3-k_1-k_2-k_3)} \delta (z^*)^{k_3} \gamma^* z^{k_2} \gamma (z^*)^{k_1} C_0^* + \cdots$$

$$+ (-1)^n (\delta \gamma^*)^{[\frac{n}{2}]} \delta^{(n-2[\frac{n}{2}])} G_{n-2[\frac{n}{2}]} \Bigg\}, \tag{217}$$

where $[\frac{n}{2}]$ represents the entire part of $\frac{n}{2}$ and $F_0 = C_0$, $F_1 = D_0^*$, $G_0 = D_0$, $G_1 = C_0^*$. Here we need to calculate the multiple summation. By expressing z as a sum of their even and odd parts, $z = z_0 + z_1$, we get, for example, $(\ell = 1, 2, \ldots, n)$

$$\sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-k_1)} \cdots \sum_{k_\ell=0}^{(n-\ell-k_1-k_2-\cdots-k_{\ell-1})} z^{(n-\ell-k_1-k_2-\cdots-k_\ell)} \gamma (z^*)^{k_\ell} \delta^* z^{k_{\ell-1}} \gamma (z^*)^{k_{\ell-2}} \delta^* \cdots$$

$$= \frac{n!}{(n-\ell)!\ell!} \Bigg\{ \overbrace{(\gamma \delta^* \gamma \delta^* \cdots)}^{\ell \text{ factors}} z_0$$

$$+ \frac{(n-\ell)}{\ell+1} \sum_{j=0}^{\ell} (-1)^{j+\ell} \overbrace{(\gamma \delta^* \gamma \delta^* \cdots)}^{(\ell-j) \text{ factors}} z_1 \overbrace{(\cdots \gamma \delta^* \gamma \cdots)}^{j \text{ factors}} \Bigg\} z_0^{(n-\ell-1)}$$

$$= \mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) z^n, \tag{218}$$

where $\mathcal{O}_{z_0}$ is the differential operator

$$\mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1) = \frac{1}{\ell!}\Bigg\{ \overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{\ell \text{ factors}} \left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right) + \frac{1}{\ell+1}\sum_{j=0}^{\ell}(-1)^{j+\ell}\overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{(\ell-j)\text{ factors}} z_1 \overbrace{(\cdots\gamma\delta^*\gamma\cdots)}^{j \text{ factors}}\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\Bigg\}, \tag{219}$$

which is also defined for $\ell = 0$, in fact $\mathcal{O}_{z_0}(0,\gamma,\delta^*,z_1) = 1$. By inserting (218) into (216) and (217), we get the compact form of C_n and D_n coefficients, i.e.,

$$C_n = \sum_{\ell=0}^{n}(-1)^\ell \mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)\frac{z^n}{\sqrt{n!}}F_{\ell-2[\frac{\ell}{2}]} \tag{220}$$

and

$$D_n = \sum_{\ell=0}^{n}(-1)^\ell \mathcal{O}_{z_0}(\ell,\delta,\gamma^*,z_1)\frac{z^n}{\sqrt{n!}}G_{\ell-2[\frac{\ell}{2}]}. \tag{221}$$

By inserting (220) and (221) into (97) and then separating the terms to multiply arbitrary constants C_0 and D_0, we obtain two independent solutions for the eigenvalue Eq. (118):

$$|\psi;-\rangle = \Bigg[\sum_{n=0}^{\infty}\sum_{\ell \text{ even}}^{2[n/2]}\mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)\frac{z^n}{\sqrt{n!}}|n;-\rangle - \sum_{n=1}^{\infty}\sum_{\ell \text{ odd}}^{2[(n+1)/2]-1}\mathcal{O}_{z_0}(\ell,\delta,\gamma^*,z_1)\frac{z^n}{\sqrt{n!}}|n;+\rangle\Bigg]C_0 \tag{222}$$

and

$$|\psi;+\rangle = \Bigg[\sum_{n=0}^{\infty}\sum_{\ell \text{ even}}^{2[n/2]}\mathcal{O}_{z_0}(\ell,\delta,\gamma^*,z_1)\frac{z^n}{\sqrt{n!}}|n;+\rangle - \sum_{n=1}^{\infty}\sum_{\ell \text{ odd}}^{2[(n+1)/2]-1}\mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)\frac{z^n}{\sqrt{n!}}|n;-\rangle\Bigg]D_0^*. \tag{223}$$

As $\mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)z^n = 0$, when $\ell > n$, we can spread out the sum on ℓ index up to infinity and then place it out of the sum corresponding to the n index. In this way, we can add up on the n index and express (222) and (223) on the form

$$|\psi;-\rangle = \Bigg[\sum_{\ell \text{ even}}^{\infty}\mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)e^{za^\dagger}|0;-\rangle - \sum_{\ell \text{ odd}}^{\infty}\mathcal{O}_{z_0}(\ell,\delta,\gamma^*,z_1)e^{za^\dagger}|0;+\rangle\Bigg]C_0 \tag{224}$$

and

$$|\psi;+\rangle = \left[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{z_0}(\ell, \delta, \gamma^*, z_1) e^{za^\dagger} |0;+\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) e^{za^\dagger} |0;-\rangle \right] D_0^*, \tag{225}$$

respectively. Finally, using the fact that $\frac{\partial^\ell}{\partial z_0^\ell} e^{za^\dagger} = (a^\dagger)^\ell e^{za^\dagger}$, we get the generalized supersqueezed states (119) and (120).

SAES OF $a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger$

Let us solve the eigenvalue Eq. (143) by taking $|\varphi\rangle$ again on the form (97). By inserting it into (143), and proceeding as in the above sections, we get the algebraic system ($n = 1, 2 \ldots$)

$$\sqrt{n+1} C_{n+1} + \gamma_0 D_n^* + \sqrt{n} \hat{\hat{\beta}}_1 C_{n-1} = z C_n, \tag{226}$$

$$\sqrt{n+1} D_{n+1} + \delta_0 C_n^* + \sqrt{n} \hat{\beta}_1 D_{n-1} = z D_n, \tag{227}$$

together with

$$C_1 = z C_0 - \gamma_0 D_0^*, \tag{228}$$

$$D_1 = z D_0 - \delta_0 C_0^*. \tag{229}$$

Again, we notice the symmetric form of this algebraic system. Proceeding by iteration, we can express the C_n and D_n coefficients in terms of the arbitrary Grassmann constants C_0 and D_0, we get ($n = 2, 3, \ldots$)

$$\begin{aligned} C_n = \tilde{C}_n - \frac{1}{\sqrt{n!}} \Bigg[& \sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-r_1)} \sum_{k_3=0}^{(n-\ell-r_2)} \cdots \sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \\ & \sum_{j=1}^{\frac{\ell}{2}} (k_{2j-1}+1) z^{(n-\ell-r_{\ell-1})} (z^*)^{k_{\ell-1}} z^{k_{\ell-2}} \cdots (z^*)^{k_1} (\sqrt{\gamma_0 \delta_0})^{\ell-2} \hat{\beta}_1 \Bigg] C_0 \\ & + \frac{1}{\sqrt{n!}} \Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]} \sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-r_1)} \sum_{k_3=0}^{(n-\ell-r_2)} \cdots \sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \\ & \sum_{j=1}^{[\frac{\ell}{2}]} (k_{2j}+1) z^{(n-\ell-r_{\ell-1})} (z^*)^{k_{\ell-1}} z^{k_{\ell-2}} \cdots z^{k_1} (\sqrt{\gamma_0 \delta_0})^{\ell-3} \gamma_0 \hat{\beta}_1 \Bigg] D_0^*, \end{aligned} \tag{230}$$

$$D_n = \tilde{D}_n - \frac{1}{\sqrt{n!}} \left[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-r_1)} \sum_{k_3=0}^{(n-\ell-r_2)} \cdots \sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \right.$$
$$\left. \sum_{j=1}^{\frac{\ell}{2}} (k_{2j-1}+1) z^{(n-\ell-r_{\ell-1})} (z^*)^{k_{\ell-1}} z^{k_{\ell-2}} \cdots (z^*)^{k_1} (\sqrt{\gamma_0 \delta_0})^{\ell-2} \hat{\beta}_1 \right] D_0$$
$$+ \frac{1}{\sqrt{n!}} \left[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]} \sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-r_1)} \sum_{k_3=0}^{(n-\ell-r_2)} \cdots \sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \right.$$
$$\left. \sum_{j=1}^{[\frac{\ell}{2}]} (k_{2j}+1) z^{(n-\ell-r_{\ell-1})} (z^*)^{k_{\ell-1}} z^{k_{\ell-2}} \cdots z^{k_1} (\sqrt{\gamma_0 \delta_0})^{\ell-3} \gamma_0 \hat{\beta}_1 \right] C_0^*, \quad (231)$$

where

$$r_\ell = \sum_{j=1}^{\ell} k_j \quad (232)$$

and, in accordance with Eqs. (220) and (221),

$$\tilde{C}_n = \sum_{\ell=0}^{n} (-1)^\ell \mathcal{O}_{z_0}(\ell, \gamma_0, \delta_0, z_1) \frac{z^n}{\sqrt{n!}} F_{\ell-2[\frac{\ell}{2}]} \quad (233)$$

and

$$\tilde{D}_n = \sum_{\ell=0}^{n} (-1)^\ell \mathcal{O}_{z_0}(\ell, \delta_0, \gamma_0, z_1) \frac{z^n}{\sqrt{n!}} G_{\ell-2[\frac{\ell}{2}]} \quad (234)$$

Using the fact that for ℓ even, we have

$$z^{(n-\ell-r_{\ell-1})} (z^*)^{k_{\ell-1}} z^{k_{\ell-2}} \cdots (z^*)^{k_1} = z_0^{(n-\ell)} + [(n-\ell)$$
$$- 2(k_1 + k_3 + \cdots k_{\ell-1})] z_0^{(n-\ell-1)} z_1, \quad (235)$$

for ℓ odd, we have

$$z^{(n-\ell-r_{\ell-1})} (z^*)^{k_{\ell-1}} z^{k_{\ell-2}} \cdots z^{k_1} = z_0^{(n-\ell)} + [(n-\ell) - 2(k_2 + k_4$$
$$+ \cdots k_{\ell-1})] z_0^{(n-\ell-1)} z_1, \quad (236)$$

and that

$$\sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-r_1)} \sum_{k_2=0}^{(n-\ell-r_2)} \cdots \sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \Lambda_\ell(k) \quad (237)$$

is equal to

$$\begin{cases} \dfrac{(n-1)!}{(n-\ell)!(\ell-1)!}, & \text{if } \Lambda_\ell(k)=1 \quad \text{and} \quad \ell \geq 2, \\ \dfrac{(n-1)!}{2(n-\ell-1)!(\ell-1)!}, & \text{if } \Lambda_\ell(k)=(k_1+k_3+\cdots+k_{\ell-1}) \\ & \quad \text{and} \quad \ell = 2,4,\ldots, \\ \dfrac{\ell(n-1)!}{2(n-\ell-1)!(\ell-1)!}\Big[(n-\ell) & \text{if } \Lambda_\ell(k)=(k_1+k_3+\cdots+k_{\ell-1})^2 \\ \quad +\dfrac{\ell}{2}(n-\ell+1)\Big], & \quad \text{and} \quad \ell = 2,4,\ldots, \\ \dfrac{(\ell-1)(n-1)!}{2(n-\ell-1)!\ell!}, & \text{if } \Lambda_\ell(k)=(k_2+k_4+\cdots+k_{\ell-1}) \\ & \quad \text{and} \quad \ell = 3,5,\ldots, \\ \dfrac{(\ell-1)(n-\ell+1)(n-1)!}{4(n-\ell-1)!\ell!}, & \text{if } \Lambda_\ell(k)=(k_2+k_4+\cdots+k_{\ell-1})^2 \\ & \quad \text{and} \quad \ell = 3,5,\ldots, \end{cases} \tag{238}$$

and after some manipulations, we can reduce (230) and (231) to

$$\begin{aligned} C_n &= \tilde{C}_n - \frac{\hat{\beta}_1}{2\sqrt{n!}} \\ &\quad \times \left[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \frac{n!}{(n-\ell)!(\ell-1)!} \left(z_0^{(n-\ell)} + \frac{(n-\ell)}{(\ell+1)} z_0^{(n-\ell-1)} z_1 \right) (\sqrt{\gamma_0\delta_0})^{\ell-2} \right] C_0 \\ &\quad + \frac{\hat{\beta}_1}{2\sqrt{n!}} \left[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} \frac{(\ell-1)n!}{(n-\ell)!\ell!} z_0^{(n-\ell)} \left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3} \gamma_0 \right] D_0^* \end{aligned} \tag{239}$$

and

$$\begin{aligned} D_n &= \tilde{D}_n - \frac{\hat{\beta}_1}{2\sqrt{n!}} \\ &\quad \times \left[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \frac{n!}{(n-\ell)!(\ell-1)!} \left(z_0^{(n-\ell)} + \frac{(n-\ell)}{(\ell+1)} z_0^{(n-\ell-1)} z_1 \right) (\sqrt{\gamma_0\delta_0})^{\ell-2} \right] D_0 \end{aligned}$$

$$+\frac{\hat{\beta}_1}{2\sqrt{n!}}\left[\sum_{\text{odd }\ell=3}^{2[\frac{n+1}{2}]-1}\frac{(\ell-1)n!}{(n-\ell)!\ell!}z_0^{(n-\ell)}(\sqrt{\gamma_0\delta_0})^{\ell-3}\delta_0\right]C_0^*, \tag{240}$$

respectively. Then, using the fact that

$$\frac{n!}{(n-\ell)!}z_0^{n-l}=\left(\frac{\partial^\ell}{\partial z_0^\ell}-z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)z^n,\qquad \frac{n!}{(n-\ell-1)!}z_0^{n-\ell-l}z_1=z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}z^n, \tag{241}$$

we can write (239) and (240) in the form

$$\begin{aligned}C_n=\tilde{C}_n-\frac{\hat{\beta}_1}{2\sqrt{n!}}&\\ \times\Bigg[\sum_{\text{even }\ell=2}^{2[\frac{n}{2}]}\frac{1}{(\ell-1)!}&\left(\left(\frac{\partial^\ell}{\partial z_0^\ell}-z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)+\frac{z_1}{(\ell+1)}\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)z^n(\sqrt{\gamma_0\delta_0})^{\ell-2}\Bigg]C_0\\ +\frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{odd }\ell=3}^{2[\frac{n+1}{2}]-1}\frac{(\ell-1)}{\ell!}&\left(\frac{\partial^\ell}{\partial z_0^\ell}-z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)(\sqrt{\gamma_0\delta_0})^{\ell-3}\gamma_0\Bigg]D_0^*\end{aligned} \tag{242}$$

$$\begin{aligned}D_n=\tilde{D}_n-\frac{\hat{\beta}_1}{2\sqrt{n!}}&\\ \times\Bigg[\sum_{\text{even }\ell=2}^{2[\frac{n}{2}]}\frac{1}{(\ell-1)!}&\left(\left(\frac{\partial^\ell}{\partial z_0^\ell}-z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)+\frac{z_1}{(\ell+1)}\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)z^n(\sqrt{\gamma_0\delta_0})^{\ell-2}\Bigg]D_0\\ +\frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{odd }\ell=3}^{2[\frac{n+1}{2}]-1}\frac{(\ell-1)}{\ell!}&\left(\frac{\partial^\ell}{\partial z_0^\ell}-z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)(\sqrt{\gamma_0\delta_0})^{\ell-3}\delta_0\Bigg]C_0^*,\end{aligned} \tag{243}$$

respectively. We notice that, when the inverse of the product $\gamma_0\delta_0$ exist, or even if it does not exist, we can write formally these last equations in the compact form

$$\begin{aligned}C_n=\tilde{C}_n-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}&\left[\sum_{\text{even }\ell=2}^{2[\frac{n}{2}]}\ell\mathcal{O}_{z_0}(\ell,\gamma_0,\delta_0,z_1)\frac{z^n}{\sqrt{n!}}\right]C_0\\ +\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}&\left[\sum_{\text{odd }\ell=3}^{2[\frac{n+1}{2}]-1}(\ell-1)\mathcal{O}_{z_0}(\ell,\gamma_0,\delta_0,z_1)\frac{z^n}{\sqrt{n!}}\right]D_0^*\end{aligned} \tag{244}$$

and

$$D_n = \tilde{D}_n - \frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\left[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \ell\mathcal{O}_{z_0}(\ell, \delta_0, \gamma_0, z_1)\frac{z^n}{\sqrt{n!}}\right] D_0$$

$$+ \frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\left[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} (\ell-1)\mathcal{O}_{z_0}(\ell, \delta_0, \gamma_0, z_1)\frac{z^n}{\sqrt{n!}}\right] C_0^*. \quad (245)$$

Now, by inserting (244) and (245) into (97) and proceeding exactly as in Appendix B (section SAES of $a + \gamma b + \delta b^\dagger$), we get the two independent solutions (145) and (146).

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